Three Essays on Operations Management Problems under Financial/Tax Consideration

XUE, Jiye

A Thesis Submitted in Partial Fulfillment

of the Requirements for the Degree of

Doctor of Philosophy

in

Decision Sciences and Managerial Economics

The Chinese University of Hong Kong

June 2015

Thesis Assessment Committee

Professor ZHU, Kaijie (Chair)

Professor HSU Vernon Ning (Thesis Supervisor)

Professor ZHOU, Xiang (Thesis Co-supervisor)

Professor EHSAN Bolandifar(Committee Member)

Professor SHEN, Zuojun (External Examiner)

ABSTRACT

Nowadays, operational financial flow has gained a growing importance in a firm’s performance, especially for some production companies. A firm lacking sufficient budget can turn to several financial schemes, such as bank credit, trade credit or inventory-based financing to support its need of working capital. For such a firm, it is worthwhile to investigate the intricate interdependence between its operational decision and financial status. Analogous to financial flow, tax consideration, such as the asymmetric tax effect at the end of the horizon should also be considered if a firm’s business is subject to this factor. Therefore we explore three studies on operations management problems under different financial/tax consideration.

In the first study, we consider a multi-period stochastic inventory control problem where a cash-constrained firm can replenish its inventory by using an inventory-based financing scheme, under which the firm is able to obtain additional capital but not exposed to default risk. We show that a state-dependent base stock policy is optimal. We partially characterize the optimal inventory and financing decisions. In particular, we show that when faced with a positive net cash level and a relatively low purchasing equity level, defined as the net cash level plus on-hand inventory valuated at the purchasing price, the firm should utilize the inventory-based financing scheme to secure more inventory. Further, when faced with a negative net cash level and a sufficiently high salvaging equity level, defined as the net cash level plus on-hand inventory valuated at the salvaging price, the firm should salvage down to a certain level. We

demonstrate, through extensive numerical experiments, that how the problem parameters may affect the optimal inventory and financing decisions and the associated profits.

In the second study, we integrate two flexibilities — quick response and trade credit in a two-level supply chain. We examine four scenarios where a manufacturer sells a seasonal product to the retailer under demand uncertainty: traditional system (T), trade-credit system (TC), quick-response system (QR), and quick-credit system which employs both quick response and trade credit (TQ). We find under both the cases where the retailer is/is not allowed to default, from the manufacturer's perspective, TC system benefits it most and under some circumstances it is worse off under TQ system than under either QR or T system; from the retailer's perspective, QR system dominates TQ system and TQ system dominates TC system, and under some cases it will benefit most under T system. In addition, by extensive numerical studies we compare the supply chain's overall expected profit under different scenarios.

In the third study, we consider a finite-horizon, discrete-time inventory control problem under tax asymmetry. The objective is to maximize the expected after-tax profit at the end of the horizon such as a tax year. We formulate the problem as a stochastic dynamic programming problem and show that a state-dependent base stock policy is optimal. We develop several structural properties that demonstrate the unique features of the proposed multiperiod inventory problem under tax consideration. We prove the fundamental insight that in each period, there exists a period-dependent equity interval, in which the firm should order less than the optimal quantity without tax asymmetry; but the firm should order the same quantity when its equity level is outside the interval. We develop some distinct analytical techniques to tackle the inherent difficulty caused by the model formulation. Our model and results can be readily adopted in some seemingly different settings such as loss aversion. Finally, we conduct numerical experiments to show several additional managerial insights.

摘要

一个公司的运营资金流对于该公司的经营业绩显得非常重要，尤其是对于一些生产性的企业。一个缺乏足够预算的中小企业可以采用不同的融资方案去筹资以支持其需要的流动资金，包括银行信贷，贸易信贷或库存质押。对于这样的公司，探讨和研究其经营决策和财务现金流状况之间的复杂的相互依存关系是非常有必要的。类似于资金流的考量，一个公司也应该考虑期末非对称的税收政策对它的营运状况带去的影响。因此在这篇论文中我们考虑三个在资金流/期末税收影响下的运营管理问题。

在第一项研究中，我们考虑一个多周期的随机库存管理问题，其中零售商在面对非平稳需求时可以通过使用库存质押的融资计划来动态地补充其库存，并且是在没有违约风险的前提下进行的。我们证明了每一期存在一个依赖状态变量的最优补货策略。进一步，我们的研究表明，当公司净现金水平为正同时购买水平（定义为净现金水平加上现有的按采购价计价的库存）相对低时，企业可以利用库存质押计划以获取更多的库存。此外，当公司净现金水平为负同时具有足够高的舍弃水平（定义为净现金水平加上现有的按舍弃价计价的库存）时，公司应该通过舍弃把库存降低到一个给定的水平。我们还通过广泛的数值试验，研究了其中的参数变化会如何影响最优补货策略和融资决策以及相关的利润变化。

在第二项研究中，我们考察了在一个二级供应链中快速反应和贸易信贷模式的结合。我们主要研究制造商在零售商面对不确定需求下的四种销售系统：传统系统（T），贸易信贷系统（TC），快速响应系统（QR），以及快速反应的信贷系统（TQ）。在允许零售商违约或者不允许零售商违约两种情况下，我们发现无论在哪种情况下从制造商的角度来看，TC系统对他最

利于而在某些情况下，TQ系统反而不如QR或者T系统；从零售商的角度来看，QR系统优于TQ系统而TQ系统优于TC系统。同时在一些情况下，T系统将使她最受惠。除此之外，我们通过数值实验比较了不同系统下的供应链整体期望利润。

在第三项研究中，我们研究一个有限期的离散时间下考虑期末非对称所得税的库存控制问题。问题的目标是期末（比如说考虑一个纳税年度之内）期望利润的最大化。作为一个随机动态规划问题，我们证明了存在一个最优的依赖状态变量的基础补货策略，并且发现了其中几个有趣的性质。比如说，在每期内存在一个权益区间，当公司的当期权益落在该权益区间内时，公司会订货到一个小于不考虑期末税收的库存水平；但当当期权益落在该区间外面时，公司会订货到一个等于不考虑税收的库存水平。我们的模型和结果可以很容易地拓展到一些其他情况，比如考虑期末带有厌恶损失的效用。最后，我们通过数值试验得到了一些其他的管理现象和启示。

Acknowledgements

First of all, I would like to show my deepest gratitude to my supervisor Prof. Vernon Hsu and co-supervisor, Prof. Sean Zhou. I am extremely lucky to have been advised by them during my Ph.D. study. I appreciate their encouragement when I get into trouble, and the enlightening and enjoyable discussion with them. In addition to my research, I also learn valuable lessons from them about all the aspects of my life.

I am also very grateful to Prof. Zhu Kaijie and Prof. Ehsan Bolandifar for being my committee members. The comments and suggestions have greatly helpful to my thesis. Likewise, I want thank many other professors, staffs and fellow visiting scholars at the Department of Decision Science and Managerial Economics for their inspiring help. I am really lucky to have so many wonderful instructions from them at CUHK.

In addition, I am very thankful to my friends at CUHK. They are Zhuang Bochao, Hu Haiju, Yu Yao, Zhang Xiaopeng, Wang Qiang, Yao Shiqing, Xu Jiayan, Yin Qianbo, Chen Zhong, Wang Tianjiao, Lei Yang, Deng Sijing, Liang Guitian, Yang Chaolin, Zhang Kairen, Wang Tong, Cui Pengxiang, Fang Tianxing, Zhong Zhiheng and Zhou Hong etc. They are all great friends and make my life here such a memorable experience.

Finally, I want to thank my father and mother, for their enormous love, encouragement and support.

Contents

List of Tables

List of Figures

CHAPTER 1

A Multi-period Inventory-based Financing Control Problem

1.1. Introduction

Due to the 2008's financial crisis and its subsequent entwined influences, many small and medium-size enterprises (SMEs) are exposed to greater risks of supply chain disruption due to limited working capital. Increasingly fluctuated market coupled with tightened bank lending policies have further worsened the plight of these SMEs, causing many of them hardly to sustain their manufacturing expenses, such as the replenishment of inventory due to insufficient cash. This has become a particularly serious problem in recent years for many SMEs in developing countries such as China. With a banking system nearly monopolized by a few state-owned banks and an immature credit reporting system, a typical SME's request for loan to sustain its working needs will often be rejected by major Chinese banks. Moreover, since many of these firms are second-tier or third-tier suppliers with relatively a weak bargaining power, it is often hard for them to obtain trade credits (a line of credit offered by a supplier to the buyer, allowing the latter to delay payment) or to secure loans using their trade accounts receivable (known as factoring). Firth et al. (2009) reported that China's credit guarantee companies served only about 1% of the country's SMEs. As a result, these SMEs often have to rely on different supply chain financial schemes, amongst which inventory-based financing is commonly adopted.

Supply chain financial schemes have been observed across different industries in many countries such as the United States and China. For example, in 2011, Bank of America supplied Barners & Noble (a large U.S. retail bookseller) with a $1 billion inventory-secured credit facility (Foley, 2012). In China, some private-owned banks have launched such financing programs in recent years. For example, Ping An Bank, a pioneer in providing supply chain financial services for China's SMEs, launched the service in 2005 and later developed online "Supply Chain Financial Platform" to provide diversified supply chain financial services including inventory-based financing. The platform reportedly attracted average daily loan of around RMB 2 billion in 2013 (Ping An Bank, 2014). Therefore, it would be interesting and also important to study firms' inventory decisions under inventory-based financing — one of the important schemes of the general supply chain financing.

Motivated by its promising perspective and potential concerns in practice, we study production and inventory management in a firm with inventory-based financing. The firm produces or acquires a single product to satisfy a stream of demands in a multi-period finite planning horizon. However, its ability to capture the demands may be constrained by its limited working capital, in which case it can use on its inventory as the collateral to secure additional loans. Clearly, faced with such a unique financing opportunity, the firm's inventory and financial flows are dynamically interdependent in complex ways. Thus, we expect that the characteristics of the firm's optimal production and inventory decisions may be very different from that under traditional supply chain settings without cash constraints. In this study, we attempt to provide managerial insights into the following questions: (a) How should a firm optimally plan for its production and inventory when faced with the opportunity of inventory-based financing? (b) What are the unique characteristics of these supply chain decisions under inventory-based financing? (c) Under what business situations will a firm significantly improve its profitability with the availability of the inventory-based financing? To address these questions, we propose a general decision framework that incorporates non-stationary demands, loan interest rate, loan-to-value ratio, and initial inventory and cash levels. In the following we summarize our main findings.

First, we find that the firm may benefit most in the situation where it is faced with a relatively low purchasing equity level, defined as the cash level plus on-hand inventory valuated at the purchasing price. Because in this case the inventory-based financing scheme helps the firm partially mitigate the negative effects of the limited cash. Second, the borrowing firm under inventory-based financing may be forced to salvage part of its inventory reactively when it is running out of cash to meet the interest payment in a period. More interestingly, however, our study suggests that the firm may proactively salvage inventory under certain conditions. For instance, when the firm has a sufficiently high salvaging equity level, defined as the cash level plus on-hand inventory valuated at the salvaging price, it may be optimal to salvage some inventory to payback all or part of the loans and bring the inventory level down to a certain level. Third, we find a main message conveyed by both our analytical and numerical studies: when the firm is short of cash and inventory, and when the predictable future demand is relatively high, it intends to over-order some inventory through inventory-based financing. This suggests that inventory-based financing may be used not only tactically, but also strategically by borrowing firms.

Our proposed research contributes to the literature by studying a new multi-period inventory management problem with inventory-based financing. Our settings differs from either the traditional settings without cash constraints or the cash-constrained settings with zero/unlimited borrowing capacity. Our study reflects real world practices of the firms that lack of credit rating or market power to rely on more traditional forms of financing such as bank credit or trade credit. Our proposed research may help these companies better understand the intricate interdependencies of inventory and financial flows and to make more informed operational and financial decisions. To the banks or other financial institutions, understanding the mechanism behind this financing scheme may help them better monitor the risks and the associated profits for their own best interests.

The rest of this article is organized as follows. In the next section, we review related literature. In Section 3, we introduce the model and characterize the general form of the optimal policy. Section 4 further studies analytical properties and the structure of the optimal policy. Section 5 presents the numerical experiments and offer additional managerial insights. Finally, this paper is concluded in Section 6.

1.2. Literature Review

Our proposed study falls into a recently active research field of the interface between supply chain management and finance. We will review parts of the rich body of literature in this field that mainly study a firm's operational and financing decisions under capital constraints. In the Operations Management literature, operational decisions of the firms that are financially constrained have been studied in recent years. One stream of this research focuses on the investigation of the impact of a firm's longer-term financial decisions, e.g., capital structure, on its operational decisions. This stream of research is motivated by uncertain and imperfect market conditions which invalidate Modigliani and Miller (1958)'s conclusion that the financial and operational decisions of a firm can be made independently. Examples of recent works in this stream include Xu and Birge (2004, 2006, 2008), Hu and Sobel (2010), and Li et al. (2013). Another stream of research in this area studies a firm which is cash constrained with operational decision relying on various short-term financing choices such as trade credit, bank credit or asset-based financing. As our work is closely related to this stream of research, we will offer a more in-depth review of papers in this stream.

Among various short-term financing choices available to a firm in the developed economies such as the United States, trade credit and bank credit are the two most popular schemes. A trade credit refers to an arrangement in which a supplier issues a line of credit to a buyer, allowing the latter to defer its payment till a certain time; a bank credit is a certain borrowing capacity a bank offers to a firm in the form of a loan with borrowing limit. To differentiate from the asset-based financing, we regards the bank credit as the financing scheme without a collateral. Many papers study various business settings under which it is advantageous to adopt a certain financing strategy that involves trade credit, bank credit or both. For example, Kouvelis and Zhao (2012) examine the optimal trade credit scheme by choosing both interest rates and wholesale price. They show under optimal trade credit contracts, both the supplier's profit and supply chain efficiency improve and the retailer may be better off compared with the bank financing scheme. Jing et al. (2012) study the financing equilibrium between trade credit financing and bank credit financing in a channel where the retailer is capital constrained. They show the production cost plays a vital role for firm's choice. Other examples include Ouyang et al. (2005), Song and Cai (2006), Zhou and Groenevelt (2007), Ho et al. (2008), Gupta and Wang (2009), Yang and Birge (2011), Caldentey and Chen (2011), Chen and Cai (2011), Kouvelis and Zhao (2011), Luo and Shang (2012), Song and Tong (2012), Chen et al. (2013), and Jing et al. (2013).

Relative to the study on operations and supply chain management with trade credit and bank credit financing, asset-based financing has drawn far less attention from academic research. Buzacott and Zhang (2004) consider a deterministic multi-period model in which a budget-constrained firm relies on asset-based financing to fund its growth. The loan limit is determined by its inventory and account receivables. The paper also considers a single-period newsvendor model in which the firm and the bank seek to maximize their own expected profit under asset-based financing. They consider the problem in a game setting where the bank determines the optimal loan interest rate and the firm decides the corresponding order quantity. Alan and Gaur (2012) study a firm's operational investment, probability of bankruptcy and capital structure under asset-based financing. They show the probability of bankruptcy and the capital structure in equilibrium are related to information asymmetry, bankruptcy cost, loan-to-value ratio, and the newsvendor model parameters.

To our knowledge, the dynamic stochastic model proposed in our paper is the first in the literature to incorporate inventory-based financing in a multi-period stochastic dynamic setting. Recently, there have been a few papers that consider inventory management of a budget-constrained firm in a multi-period stochastic demand setting. Chao et al. (2008) analyze the optimal inventory policy of a budget-constrained firm. The firm can only acquire inventory under the budget-limit and cannot borrow from the bank. They show the dependence of the firm's optimal inventory policy on its financial status. Gong et al. (2011) extend this work by allowing the firm to borrow from a bank with an unlimited capacity. They characterize the optimal inventory policy under different "equity" level through both analytical and numerical studies. Our inventory-based financing model is different from these two works. Indeed, the borrowing capacity of the firm in our proposed study is between that of Chao et al. (2008)

(zero capacity) and Gong et al. (2014) (unlimited capacity). In addition, this borrowing capacity is strongly tied to the firm's inventory decision, a relationship that is not considered in the previous two papers. Finally and connected to the previous point, we consider the salvaging decision in each period, allowing the firm to dispose some excess inventory to repay all or part of the current outstanding loan.

1.3. Model

In this paper, we consider a manufacturer/retailer facing a multi-period inventory-based financing problem. Compared to the traditional inventory management problem, the firm only has limited funds for production/procurement. However, it can collateralize some of its inventory to get loan from a monetary party such as a bank. Specifically, the sequence of events in each period are as follows: (1) Determine the salvage quantity and pay interests for loan carried to this period; (2) Borrow or pay back the outstanding loan, subject to the available inventory and cash, pay custody fee for the collateralized inventory; (3) Determine production quantity, and the production takes place instantaneously; (4) Demand is realized and satisfied as much as possible; (5) Unmet demand is lost, leftover inventory is carried to the next period.

Remark. Using inventory finance, as a common practice, the manufacturer needs to have its collateralized inventory held and monitored by a third party, such as a third-party logistics service provider, which incurs a relatively high custody fee or holding costs. For simplicity, we assume that holding costs for the normal inventory held by the manufacturer itself is ignorable. This is reasonable, because in practice this costs is relatively low compared with the custody fee and can be counted as overhead costs.

Define for $t = 1, 2, \cdots, T+1$,

t = period index,

$\hat{U}_t$ = initial cash in period t,

x_t = initial inventory level in period t,

$\hat{q}_t$ = production quantity in period t,

z_t = salvage quantity in period t,

D_t = random demand in period t, D_t can be any non-stationary random variable,

$\hat{l}_t$ = initial loan in period t,

b_t = net borrowing in period t, $b_t < 0$ represents a loan repayment,

$\hat{l}_t + b_t$ = loan in period t after borrowing/repayment, it is also the initial loan in period $t+1$, i.e., $\hat{l}_{t+1} = \hat{l}_t + b_t$.

In addition, define

p = unit sales price,

c = unit procurement price,

s = unit salvage value at anytime,

h = unit custody fee of collateralized inventory,

α' = loan interest rate charged by the bank,

γ = proportion of the salvage value of the collateralized inventory that the manufacturer can borrow, also known as the loan-to-value ratio and it obviously follows $0 \le \gamma \le 1$.

Note here we do not consider the deposit interest rate since we do not tend to over complicate the financial flows within the firm. Thus we only focus on the cash outflow when the firm borrows from the bank. Suppose the manufacturer collateralizes one unit of inventory at the beginning of a period, it can obtain a loan with the amount of γs. To make sure the manufacturer is able to pay back interest and principal, the following condition must hold:

$$s - h - (1 + \alpha')\gamma s \geq 0.$$

This is because the total cost includes interest, principal and holding cost. The above inequality is equivalent to define an "overall" interest rate:

$$\alpha \triangleq \alpha' + \frac{h}{\gamma s}$$

and this new interest rate should satisfy

$$\alpha = \alpha' + \frac{h}{\gamma s} \leq \frac{s}{\gamma s} - 1 = \frac{1}{\gamma} - 1.$$

In the following discussion, we shall still call α "interest rate" and the associated costs "interests".

The dynamic transitions can be expressed as follows. For $t = 1, \cdots, T$,

$$\begin{aligned}
x_{t+1} &= (x_t - z_t + \hat{q}_t - D_t)^+, \\
\hat{l}_{t+1} &= \hat{l}_t + b_t, \\
\hat{U}_{t+1} &= (\hat{U}_t - \alpha \hat{l}_t + s z_t + b_t - c\hat{q}_t) + p \min\{D_t, x_t - z_t + \hat{q}_t\}.
\end{aligned}$$

Furthermore, we have the following constraints,

$$c\hat{q}_t \leq \hat{U}_t - \alpha\hat{l}_t + b_t + sz_t, \tag{1.1}$$

$$\hat{l}_t + b_t \leq \gamma s(x_t - z_t), \tag{1.2}$$

$$\hat{l}_t + b_t \geq 0, \tag{1.3}$$

$$\hat{U}_t - \alpha\hat{l}_t + b_t + sz_t \geq 0, \tag{1.4}$$

$$\hat{q}_t \geq 0, 0 \leq z_t \leq x_t, \tag{1.5}$$

$$\hat{U}_1 - (1+\alpha)\hat{l}_1 + sx_1 > 0. \tag{1.6}$$

Constraint (1.1) guarantees the production cost is no more than the available cash. Constraint (1.2) ensures that the firm should not have its total debt more than its collateral evaluated by the loan-to-value ratio. This constraint guarantees a risk-free position to the bank, which also means the firm will never "default" for the on-going business. Technically, we exclude the case in which the dynamic process will terminate at a point when the firm is not able to payback the outstanding loan via its total asset (includes both inventory and cash). Note that we assume fixed salvage value of the collateral, which helps guarantee the risk-free environment and makes the problem tractable. Introducing a varied salvage value seems more realistic but it will inevitably bring technical challenges and we leave it to future work. Practically, the "never-default" assumption is reasonable and reflects many industry practices of supply chain financing we observe in China. For example, based on public data, the default rate of the loans based on supply chain financing of Ping An Bank during 2005–2009 is only 0.65%, which is fairly small compared with 5.7% default rate for the regular loans during the same period, thereby empirically supporting our "never-default" assumption. In fact, in case of inventory-based financing, often, an authorized party such as a third-party logistics provider continuously monitors the market value of the collateral and tries to ensure that the borrowing firm is able to payback the outstanding loan. Constraint (1.3) says that the net loan is nonnegative; constraint (1.4) makes sure that the repayment is no more than the available cash. Furthermore, to make the model meaningful, we assume the manufacturer has some initial inventory and/or net cash, as expressed in inequality (1.6).

Let $V_t(x_t, \hat{U}_t, \hat{l}_t)$ = maximum profit-to-go starting from period t with initial inventory x_t, initial cash $\hat{U}_t$, and initial loan $\hat{l}_t$. We can write the optimality equations as follows.

For $t = T + 1$:

$$V_{T+1}(x_{T+1}, \hat{U}_{T+1}, \hat{l}_{T+1}) = \hat{U}_{T+1} - (1+\alpha)\hat{l}_{T+1} + sx_{T+1};$$

For $t = 1, \cdots, T$:

$$\begin{aligned} V_t(x_t, \hat{U}_t, \hat{l}_t) \quad = \quad & \max_{\substack{(z_t, \hat{q}_t, b_t) \text{ satisfies} \\ \text{inequalities (1.1) -- (1.5)}}} \mathsf{E}\Big[V_{t+1}((x_t - z_t + \hat{q}_t - D_t)^+, \\ & (\hat{U}_t - \alpha\hat{l}_t + sz_t + b_t - c\hat{q}_t) + p\min\{D_t, x_t - z_t + \hat{q}_t\}, \hat{l}_t + b_t)\Big]. \end{aligned}$$

Using arguments similar to Chao et al. (2008), we can prove Lemmas 1 and 2 and Proposition 1.

Lemma 1.3.1. *For fixed x_t and $\hat{l}_t$, $\hat{V}_t(x_t, \hat{U}_t, \hat{l}_t)$ is increasing in $\hat{U}_t$.*

Lemma 1.3.2. $\hat{V}_t(x_t, \hat{U}_t, \hat{l}_t) \geq \hat{V}_t(x_t + \Delta, \hat{U}_t - p\Delta, \hat{l}_t)$ *for any* $\Delta \geq 0$.

Note that $\Delta \geq 0$ is essential for Lemma 1.3.2, for otherwise if $\Delta < 0$, then

$\hat{l}_t$ may violate the γ ratio on the loan. The next property shows that the value function is jointly concave.

Proposition 1.3.1. *$\hat{V}_t(x_t, \hat{U}_t, \hat{l}_t)$ is jointly concave in $(x_t, \hat{U}_t, \hat{l}_t)$ for $t = 1, \cdots, T+1$.*

Proof. See Appendix. $\square$

Proposition 1.3.2. *There is an optimal policy such that salvaging and production may not occur simultaneously, namely, we have (1) $z_t > 0$ only if $\hat{q}_t = 0$, and (2) $\hat{q}_t > 0$ only if $z_t = 0$.*

Proof. Suppose we have an optimal solution such that in some period t_0, we have $z_{t_0} > 0$ and $\hat{q}_{t_0} > 0$. We can modify the solution by reducing salvaging quantity to $z_{t_0} - \min\{z_{t_0}, \hat{q}_{t_0}\}$ and production quantity to $\hat{q}_{t_0} - \min\{z_{t_0}, \hat{q}_{t_0}\}$. It is easy to check the the solution is feasible and by Lemma 1.3.1 it is also a better solution with increased (at least equally good) expected profit. $\square$

By Proposition 2, the two decision variables z_t and $\hat{q}_t$ may be replaced by one, so we define $q_t = \hat{q}_t - z_t$. To further simplify the problem, we define two new state variables, $U_t = \hat{U}_t - (1+\alpha)\hat{l}_t$ and $l_t = \hat{l}_t + b_t$. By definition, q_t means the net ordering quantity and U_t represents the net cash level, which may be negative at the beginning of period t as if the firm has paid back all the outstanding loan. Also, l_t represents the total outstanding loan after borrowing or repayment.

Then for $t = 1 \cdots, T$, we have the following new constraints,

$$
\begin{aligned}
x_{t+1} &= (x_t + q_t - D_t)^+, \\
U_{t+1} &= (U_t + sq_t^- + l_t - cq_t^+) + p\min\{D_t, x_t + q_t\} - (1+\alpha)l_t, \\
cq_t^+ &\le U_t + sq_t^- + l_t && (1.7)\\
l_t &\le \gamma s x_t, && (1.8)\\
l_t &\le \gamma s(x_t + q_t), && (1.9)\\
l_t &\ge 0, && (1.10)\\
q_t &\ge -x_t, && (1.11)\\
U_1 + sx_1 &> 0. && (1.12)
\end{aligned}
$$

Let $V_t(x_t, U_t)$ = maximum profit-to-go starting from period t with initial inventory x_t and initial net cash U_t. Then we can write the optimality equations as follows:

For $t = 1, \cdots, T$:

$$
\begin{aligned}
V_t(x_t, U_t) = \max_{\substack{(l_t, q_t) \text{ satisfies} \\ \text{inequalities } (1.7)-(1.11)}} \mathsf{E}\Big[& V_{t+1}((x_t + q_t - D_t)^+, (U_t + sq_t^- + l_t - cq_t^+) \\
& + p\min\{D_t, x_t + q_t\}) - (1+\alpha)l_t \Big].
\end{aligned}
$$

For $t = T + 1$:

$$
V_{T+1}(x_{T+1}, U_{T+1}) = U_{T+1} + sx_{T+1}.
$$

The following result can be proved in a way similar to the proof of Proposition 1.3.1.

Proposition 1.3.3. *$V_t(x_t, U_t)$ is jointly concave in (x_t, U_t) for $t = 1, \cdots, T+1$.*

In addition, it is easy to verify the following property.

Proposition 1.3.4. *For a feasible production (salvage) quantity q_t in period t, the optimal outstanding loan l_t in that period can be expressed as follows:*

$$l_t = \begin{cases} 0, & \text{if } U_t + sq_t^- - cq_t^+ \geq 0, \\ cq_t^+ - U_t - sq_t^-, & \text{if } U_t + sq_t^- - cq_t^+ < 0. \end{cases} \tag{1.13}$$

The above proportion has an intuitive interpretation: if there is a positive outstanding loan at some period t, then the manufacturer must have used all the available cash including the loan in that period. For otherwise it can use the leftover cash to payback part of the loan because of the loan interest will incur in the next period. Because

$$\mathsf{E}\Big[V_{t+1}((x_t + q_t - D_t)^+, (U_t + sq_t^- + l_t - cq_t^+) + p\min\{D_t, x_t + q_t\} - (1+\alpha)l_t)\Big]$$

is jointly concave in (x_t, U_t, q_t, l_t), $V(x_t, U_t)$ can be obtained by sequential maximization over l_t and q_t, so

$$
\begin{aligned}
& V_t(x_t, U_t) \\
&= \max_{-x_t \le q_t \le \min\{\frac{U_t+\gamma s x_t}{c}, \frac{U_t+\gamma s x_t}{(1-\gamma)s}\}} \max_{\substack{l_t \text{ satisfies} \\ \text{inequalities (1.7) -- (1.11)}}} \mathsf{E}\Big[V_{t+1}((x_t+q_t-D_t)^+, \\
&\qquad (U_t + sq_t^- + l_t - cq_t^+) + p\min\{D_t, x_t+q_t\} - (1+\alpha)l_t)\Big] \\
&= \max_{-x_t \le q_t \le \min\{\frac{U_t+\gamma s x_t}{c}, \frac{U_t+\gamma s x_t}{(1-\gamma)s}\}} \mathsf{E}\Big[V_{t+1}((x_t+q_t-D_t)^+, \\
&\qquad (U_t + sq_t^- - cq_t^+)^+ + p\min\{D_t, x_t+q_t\} - (1+\alpha)(cq_t^+ - U_t - sq_t^-)^+)\Big], \\
&= \max_{0 \le y_t \le \min\{x_t + \frac{U_t+\gamma s x_t}{c}, x_t + \frac{U_t+\gamma s x_t}{(1-\gamma)s}\}} \mathsf{E}\Big[V_{t+1}((y_t-D_t)^+, (U_t + s(y_t-x_t)^- - c(y_t-x_t)^+)^+ \\
&\qquad + p\min\{D_t, y_t\} - (1+\alpha)(c(y_t-x_t)^+ - U_t - s(y_t-x_t)^-)^+)\Big],
\end{aligned}
$$

where the second equality is due to (1.13) and the third equality is obtained by letting the order-up-to level $y_t = x_t + q_t$. The constraints of q_t deserve some explanations. First, the inequality $q_t \ge -x_t$ is from (1.11) and ensures that the firm cannot salvage more than its on-hand inventory. Second, if it occurs the case $U_t + \gamma s x_t < 0$, which means the cash obtained from collateralizing all available inventory cannot cover all the outstanding loan carried to this period, then from (1.9) and (1.13), we can see that the firm has to salvage some of its on-hand inventory to payback part of the loan. The minimum salvage quantity, denoted q_t^1, is determined by the break-even condition

$$U_t + \gamma s(x_t + q_t^1) - sq_t^1 = 0.$$

Thus, the firm has to salvage at least $q_t^1 = \frac{U_t+\gamma s x_t}{s-\gamma s} < 0$ in order to guarantee that outstanding loan can be covered by the collateral. In this case, define a new state variable $x_t^1 = x_t + q_t^1$ and $U_t^1 = U_t - sq_t^1$ after salvaging and note $U_t^1 + \gamma s x_t^1 \geq 0$, which eventually meets the collateral requirement in (1.9). Finally, we impose the constraint $q_t \leq \frac{U_t + \gamma s x_t}{c}$ based on (1.7) and (1.13) to meet the budget constraint when the firm orders instead of salvaging inventory.

Remark. From the above discussions, when $U_t + \gamma s x_t < 0$, we can modify the state variables to obtain an equivalent problem as follows: $x_t^1 = x_t + q_t^1$ and $U_t^1 = U_t - sq_t^1 \leq 0$, where $q_t^1 = \frac{U_t + \gamma s x_t}{(1-\gamma)s} < 0$. This means that the firm should first salvage some inventory and the resulting collateral constraint becomes $U_t^1 + \gamma s x_t^1 = 0$ and the optimality equation becomes

$$\begin{aligned} V_t(x_t^1, U_t^1) \quad = \quad & \max_{-x_t \leq q_t \leq q_t^1} \mathsf{E}\Big[V_{t+1}((x_t + q_t - D_t)^+, (U_t^1 + s(x_t + q_t - x_t^1)^- \\ & - c(x_t + q_t - x_t^1)^+)^+ + p\min\{D_t, y_t\} - (1+\alpha)(c(x_t + q_t - x_t^1)^+ \\ & - U_t - s(x_t + q_t - x_t^1)^-)^+)\Big], \end{aligned}$$

or equivalently

$$\begin{aligned} V_t(x_t^1, U_t^1) \quad = \quad & \max_{0 \leq y_t \leq x_t^1} \mathsf{E}\Big[V_{t+1}((y_t - D_t)^+, (U_t^1 + s(y_t - x_t^1)^- - c(y_t - x_t^1)^+)^+ \\ & + p\min\{D_t, y_t\} - (1+\alpha)(c(y_t - x_t^1)^+ - U_t - s(y_t - x_t^1)^-)^+)\Big], \end{aligned}$$

where $U_t^1 < 0$ and $U_t^1 + \gamma s x_t^1 \geq 0$. Therefore, in the rest of the paper, unless otherwise specified, we will assume without loss of generality that the state variables satisfy $U_t + \gamma s x_t \geq 0$.

Based on the above discussions, we obtain the following result.

Proposition 1.3.5. *The optimal policy is a state-dependent base stock policy. Specifically, in each period t, there is an optimal order-up-to or salvage-down-to level $\hat{y}^*(x_t, U_t)$ and an associated optimal outstanding loan level $l^*(x_t, U_t)$ such* *that if an ordering decision is made, then order up to $\hat{y}_t^*(x_t, U_t)$ otherwise if a salvaging decision is made, then salvage down to $\hat{y}_t^*(x_t, U_t)$. The associated optimal loan level is expressed by Equation (1.13).*

Let y_t^* be the optimal base-stock level in period t and recall the optimality equation is

$$V_t(x_t, U_t) = \max_{0 \le y_t \le \min\{x_t + \frac{U_t + \gamma s x_t}{c}, x_t + \frac{U_t + \gamma s x_t}{(1-\gamma)s}\}} \mathsf{E}\Big[V_{t+1}((y_t - D_t)^+, (U_t + s(y_t - x_t)^- \\ -c(y_t - x_t)^+)^+ + p\min\{D_t, y_t\} - (1+\alpha)(c(y_t - x_t)^+ - U_t - s(y_t - x_t)^-)^+)\Big].$$

To further exploit the problem structures, we define $Q_t = U_t + sx_t$ and $R_t = U_t + cx_t$. Q_t and R_t can be interpreted as two kinds of equity levels: salvaging equity level and purchasing (or production) equity level, which are the cash level U_t plus the inventory x_t valuated at the salvaging value s and the production cost c, respectively. Incorporating these two equity levels, we define three value functions:

$$\Pi_t^s(y_t, Q_t) = \mathsf{E}\Big[V_{t+1}((y_t - D_t)^+, p\min\{D_t, y_t\} + (1+\alpha)(Q_t - sy_t))\Big],$$

$$\Pi_t^d(y_t, R_t) = \mathsf{E}\Big[V_{t+1}((y_t - D_t)^+, p\min\{D_t, y_t\} + (R_t - cy_t))\Big],$$

$$\Pi_t^{br}(y_t, R_t) = \mathsf{E}\Big[V_{t+1}((y_t - D_t)^+, p\min\{D_t, y_t\} + (1+\alpha)(R_t - cy_t))\Big].$$

$\Pi_t^s(y_t, Q_t)$ can be interpreted as the value function when the firm decides to salvage inventory but still cannot fully repay the total outstanding loan. Similarly, $\Pi_t^d(y_t, R_t)$ is the value function when the firm orders by its net cash and

$\Pi_t^{br}(y_t, R_t)$ is the value function when the firm makes a production decision by borrowing under inventory-based financing. Given Q_t and R_t, similar to Proposition 2, we can prove Π_t^s, Π_t^d, and Π_t^{br} are all concave in y_t and denote $y_t^s(Q_t)$, $y_t^d(R_t)$ and $y_t^{br}(R_t)$ are the maximizers over the constraint $y_t \geq 0$, respectively.

1.4. Optimal Policy Structure

In this section, we shall further investigate several properties related to the optimal policy. Recall that we assume without loss of generality that the collateral requirement $U_t + \gamma s x_t \geq 0$ holds at the beginning of period t. We consider two cases, namely, $U_t \geq 0$ and $U_t \leq 0$, in the following two subsections.

1.4.1. Nonnegative Initial Cash Level ($U_t \geq 0$)

When the firm has some initial positive net cash at the beginning of period t (i.e., $U_t \geq 0$), it is easy to prove the following property.

Lemma 1.4.1. *If $U_t \geq 0$, then the firm should never salvage any inventory in this period.*

This is straightforward because under the conditions of zero holding cost and random demand, salvaging current inventory is no better than keeping all the inventory on hand when there is no initial outstanding loan.

Lemma 1.4.2. *For any period t, $t = 1, \cdots, T$, the following relationship is satisfied:*
$\hat{y}_t^* = y_t^d$ and $y_t^* = \max\{y_t^d, x_t\}$, if $y_t^d \leq R_t/c$; otherwise $\hat{y}_t^* = y_t^* = R_t/c$, if $y_t^{br} \leq R_t/c \leq y_t^d$; otherwise $\hat{y}_t^* = y_t^{br}$ and $y_t^* = min\{y_t^{br}, \frac{R_t+\gamma s x_t}{c}\}$, if $R_t/c \leq y_t^{br}$.

Proof. The proof of this lemma is straightforward by using the first order condition. □

Before further analysis, we introduce a sequence of concave functions $G_t^d(y_t)$ as follows: for $t = 1, \cdots, T$,

$$G_t^d(y_t) = \mathsf{E}\Big[(p-c)\min\{D_t, y_t\} + G_{t+1}^d(\max\{a_{t+1}^d, (y_t - D_t)^+\})\Big],$$

and $G_{T+1}^d(y_t) = (s-c)y_t$ for $t = T+1$, where $a_{T+1}^d = 0$ and a_t^d is the maximizer of the concave function $G_t^d(y_t)$ for $t = 1, \cdots, T$. Suppose $G_{t+1}^d(y_t)$ is concave and we have $\frac{d^2 G_t^d(y_t)}{dy_t^2} = (c-p)f_t(y_t) + \mathsf{E}_{\{y_t - D_t \geq a_{t+1}^d\}}[\frac{d^2 G_{t+1}^d(y_t - D_t)}{dy_t^2}]$. Note $p > c$ and $\frac{d^2 G_{t+1}^d(y_t - D_t)}{dy_t^2} \leq 0$ when $y_t - D_t \geq a_{t+1}^d$, it holds $\frac{d^2 G_t^d(y_t)}{dy_t^2} \leq 0$, which guarantees the concavity of $G_t^d(y_t)$. Based on $G_t^d(y_t)$ and a_{t+1}^d, we construct another two sequences of concave functions $G_t^s(y_t)$ and $G_t^{br}(y_t)$ in the following, for $t = 1, \cdots, T$,

$$G_t^s(y_t) = \mathsf{E}\Big[(p-c)\min\{D_t, y_t\} + cy_t - (1+\alpha)sy_t + G_{t+1}^d(\max\{a_{t+1}^d, (y_t - D_t)^+\})\Big],$$

$$G_t^{br}(y_t) = \mathsf{E}\Big[(p-c)\min\{D_t, y_t\} - \alpha c y_t + G_{t+1}^d(\max\{a_{t+1}^d, (y_t - D_t)^+\})\Big].$$

Define a_t^s and a_t^{br} as the maximizers of $G_t^s(y_t)$ and $G_t^{br}(y_t)$. The construction above is analogous to Gong et al. (2011), in which they introduce functions $G_t^d(y_t)$ and $G_t^{br}(y_t)$ as the major devices to characterize the policy structure. The main differences between their approach and ours are that we introduce the salvage function $G_t^s(y_t)$ to reflect the firm's salvage decisions and that we assume a non-stationary demand distribution. In addition, the limited borrowing capacity due to inventory-based financing adds more complex technical issues and also generates different policy structures.

Lemma 1.4.3. *The following relationship regarding the parameters a_t^s, a_t^d and a_t^{br} is satisfied: For $t = 1, \cdots, T$, (i) $a_t^s \geq a_t^d \geq a_t^{br}$; (ii) if $a_{t+1}^d \leq F_{t+1}^{-1}(\frac{p-(1+\alpha)c}{p-c})$, then $a_t^{br} \geq a_{t+1}^d$; otherwise, $a_t^{br} = F_t^{-1}(\frac{p-(1+\alpha)c}{p-c})$.*

Proof. It is easy to verify that $\frac{dG_t^s(y_t)}{dy_t} = \frac{dG_t^d(y_t)}{dy_t} + (c - \alpha s - s)$ and $\frac{dG_t^s(y_t)}{dy_t} = \frac{dG_t^d(y_t)}{dy_t} - \alpha c$, which immediately implies $a_t^s \geq a_t^d \geq a_t^{br}$. The proof of the second part is referred to (iii) of Theorem 2 in Gong et al. (2011). □

In the next proposition, we characterize the optimal inventory policy for three different regions.

Proposition 1.4.1. *For $t = 1, \cdots, T$, suppose the initial cash $U_t \geq 0$, the following results hold:*

(i) when $R_t \geq ca_t^d$, the optimal policy is an order-up-to policy with $\hat{y}_t^* = a_t^d$ and $y_t^* = \max\{x_t, a_t^d\}$;

(ii) when $ca_t^{br} \leq R_t \leq ca_t^d$, the optimal policy is an order-up-to policy with $\hat{y}_t^* = y_t^* = R_t/c$ by using up all on-hand cash;

(iii) when $R_t \leq ca_t^{br}$, the optimal policy is a state-dependent order-up-to policy with $\hat{y}_t^* = y_t^{br}$ and $y_t^* = \min\{y_t^{br}, \frac{R+\gamma s x_t}{c}\}$ by borrowing under inventory-based financing.

Proof. See Appendix. □

From this proposition, we see that in the case of a nonnegative initial cash the equality level plays an important role in determining the optimal policy. Specifically, we can see when the purchasing equity level R_t is above a threshold

a_t^d, due to the decomposition property of the value function $\Pi_t^d(y_t, R_t)$, the firm should optimally order up to a constant inventory level a_t^d; for a smaller $R_t \in [ca_t^{br}, ca_t^d]$, the optimal policy is to order up to the level R_t/c by simply using up all the on-hand cash; when R_t further decreases to a level less than ca_t^{br}, the firm will turn to inventory-based financing for "over-ordering". This behavior occurs when the current on-hand capital is insufficient to afford the inventory for current or future demand. Compared to the traditional inventory control setting without budget constraints, the "poor" firm in our model can not order up to that unconstrained level a_t^d. In addition, we see that the optimal order-up-to level under the inventory-based financing is between the one under zero capacity in Chao et al. (2008) and the one under unlimited capacity in Gong et al. (2014). The following example illustrates the structure of the optimal policy.

Example 1.4.1. *Consider a two-period instance with* $p = 1.3$, $c = 1$, $s = 0.5$, $\gamma = 0.5$, $\alpha = 0.1$, $D_1 \in [0, 10] \sim$ *truncated normal distribution with mean 5 and standard deviation 5,* D_2 *has the same distribution as* D_1 *and* D_2 *is independent of* D_1*. The initial cash level in period 1 is* $U = 2$*. The optimal policy for different initial inventory level* $x \in [0, 10]$ *is shown in Figure 1.1.*

Figure 1 shows that when the initial cash level is positive and when the inventory level is low, the firm should over-order by using more than its on-hand cash through inventory-based financing. When the inventory level increases, it should first use up its on-hand cash and then order to a certain level by using part of its cash or even not order. Note that as stated in Lemma 3, in this example the initial cash level is positive, so the firm should not consider salvaging existing inventory in the current period.

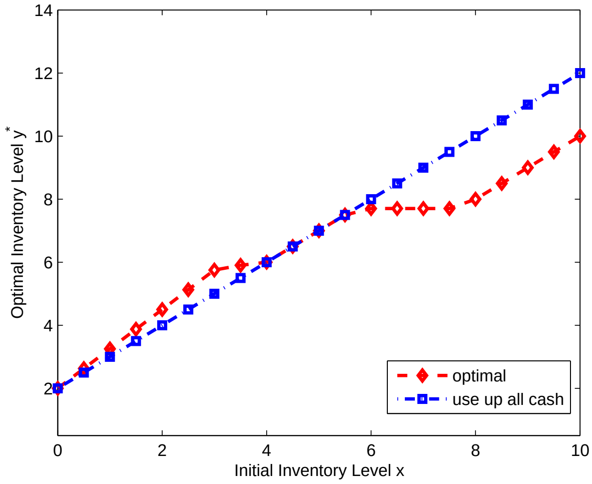

Figure 1.1: Optimal Inventory Level for Different x with $U = 2$

1.4.2. Negative Initial Cash Level ($U_t < 0$)

With some outstanding loans at the beginning of period t (i.e., $U_t < 0$), in addition to ordering, the firm could also instead consider the salvaging option to pay back part or all of the loan if it has a large amount of inventory. We first provide a property that characterizes the optimal policy when its salvaging equity level Q_t is sufficiently high and then discuss some examples to illustrate the structures and complexities in this case.

Proposition 1.4.2. *For $t = 1, \cdots, T$, suppose $U_t < 0$, the following property holds: when $Q_t \geq sa_t^s$, the optimal policy is a salvage-down-to policy with $\hat{y}_t^*(Q_t) = y_t^*(Q_t) = Q_t/s$;*

Proof. See Appendix. □

We see that the firm optimally pays back all the outstanding loan by salvaging some inventory when its salvaging equity level is above a threshold. For other cases that the salvaging equity level is not this high, due to the complexity of whether to salvage or to order, it seems unlikely we can decompose the value function in a manner similar to the previous analysis. In the following we will first illustrate the general structure of the optimal policy via a two-period example, and then further discuss the structure of a two-period special case with deterministic demand in the first period and random demand in the second period.

Example 1.4.2. *Consider a two-period instance with* $p = 1.3$*,* $c = 1$*,* $s = 0.5$*,* $\gamma = 0.5$*,* $\alpha = 0.1$*,* $D_1 \sim [0, 10]$ *follows a truncated normal distribution with mean 5 and standard deviation 5;* D_2 *has the same distribution as* D_1 *and* D_2 *is independent of* D_1*. The initial cash level in period 1 is* $U = -2$*. The optimal policy under different initial inventory level* $x \in [5, 24]$ *is shown in Figure 1.2.*

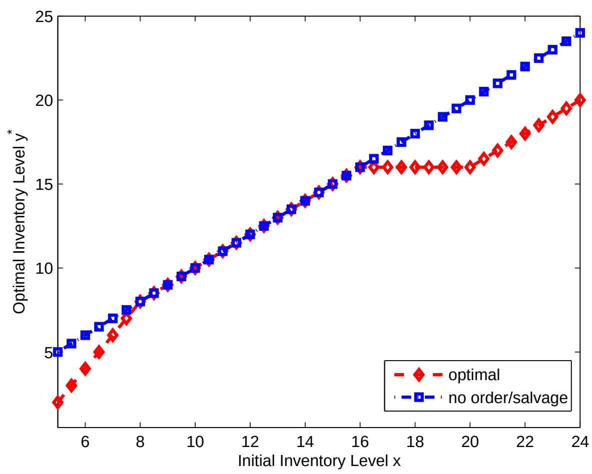

Figure 1.2: Optimal Inventory Level for Different x with $U = -2$

Figure 2 provides us a concrete picture of how the optimal policy behaves when the firm has some initial outstanding loan. We observe the optimal inventory level y^* increases in the initial inventory level x given $U = -2$. We further find four intervals regarding the optimal inventory level y^* for different

x: when x is small, i.e., $x \in [5, 8]$, y^* lies below x which means the firm should salvage some inventory to reach the collateral requirement; with a larger x, i.e., $x \in [8, 16]$, the firm should stay at the initial inventory level and y^* should be equal to x; when x further increases, i.e., $x \in [16, 21]$, the firm should salvage down to a certain level $y^* = 16$; for the rest of x, i.e., $x \in [21, 24]$, the firm should salvage the inventory to pay back all the outstanding loan, which is asserted in Proposition 1.4.2. Finally, we note the firm should not order any additional inventory in this example. This is understandable since the firm will incur more interests and yet barely get profits from the additional inventory when it chooses to over-order.

Next, to better understand the firm's ordering and salvaging decisions, we consider a two-period special case with the following settings: the demand d in the first period is deterministic and the demand D in the second period is random with distribution function $F(\cdot)$ and support $[0, \infty)$. This special case seems plausible because the current demand is often much easier to estimate than the future trends. Suppose the initial cash level and inventory level in the first period are x and U, respectively. We focus on the case where $U < 0$ and $x > d$; other cases can be analyzed through the procedures in Section 4.1. Without loss of generality we assume $U + \gamma s x \geq 0$. Then, the firm's decision problem in the first period can be described as follows:

$$\begin{aligned} V_1(x, U) \;\; = \;\; & \max_{0 \leq y \leq x + \frac{U + \gamma s x}{c}} V_2\big((y-d)^+, p \min\{y, d\} + (U + s(x-y)^+ - c(y-x)^+)^+ \\ & \qquad - (1+\alpha)(c(y-x)^+ - U - s(x-y)^+)^+\big), \end{aligned}$$

where

$$
\begin{aligned}
& V_2(x_2, U_2) \\
= & \max_{0 \le y_2 \le x_2 + \min\{\frac{U_2+\gamma s x_2}{c}, \frac{U_2+\gamma s x_2}{(1-\gamma)s}\}} \mathsf{E}\Big[p \min\{y_2, D\} + (U + s(x_2 - y_2)^+ - c(y_2 - x_2)^+)^+ \\
& \qquad - (1+\alpha)(c(y_2 - x_2)^+ - U_2 - s(x_2 - y_2)^+)^+ + s(y_2 - D)^+\Big].
\end{aligned}
$$

We assume that $0 \le \frac{\alpha\gamma s + \alpha s + \gamma s - \gamma s^2/c}{(p-s)(\gamma s - \alpha c)/c} \le 1$ (otherwise the firm will use a simpler order-up-to policy), and let z^0 be the unique solution to $\bar{F}(\frac{pd+(1+\alpha)(R-cd)+(\gamma s-\alpha c)z}{c}) = \frac{\alpha\gamma s+\alpha s+\gamma s-\gamma s^2/c}{(p-s)(\gamma s-\alpha c)/c}$.

Proposition 1.4.3. *For the special case defined above, assume $U < 0$ and $x > d$, and $0 \le \frac{\alpha\gamma s+\alpha s+\gamma s-\gamma s^2/c}{(p-s)(\gamma s-\alpha c)/c} \le 1$. If $z^0 + d \ge x$, the optimal policy is the order-up-to policy with $\hat{y}^* = \min\{z^0 + d, \frac{R+\gamma s x}{c}\}$; otherwise the firm should salvage down to $\hat{y}^*$ that can be expressed as follows:*

$$
\hat{y}^* = \begin{cases}
\frac{pd+(1+\alpha)Q}{(1+\alpha)s}, & z^1 \le \frac{(p-s-\alpha s)d+(1+\alpha)Q}{(1+\alpha)s}; \\
z^1 + d, & \frac{(p-s-\alpha s)d+(1+\alpha)Q}{(1+\alpha)s} \le z^1 \le \frac{(p-s-\alpha s)d+(1+\alpha)Q}{(1+\alpha)s-\gamma s}; \\
\frac{(p-\gamma s)d+(1+\alpha)Q}{(1+\alpha)s-\gamma s}, & z^1 \ge \frac{(p-s-\alpha s)d+(1+\alpha)Q}{(1+\alpha)s-\gamma s},
\end{cases}
$$

where $z^1 = F^{-1}(\frac{p-(1+\alpha)^2 s}{p-s})$. In addition, the optimal inventory level $y^* = \min\{x, \hat{y}^*\}$.

Proof. See Appendix. □

We see two opposite decisions for different initial inventory levels in this case. On the one hand, with relatively low inventory level and anticipating a high future demand, the firm would acquire additional inventory through inventory-based financing. In fact, the "ordering" condition $z^0 + d \ge x$, i.e.,

$c\bar{F}^{-1}\big(\frac{\alpha\gamma s+\alpha s+\gamma s-\gamma s^2/c}{(p-s)(\gamma s-\alpha c)/c}\big) - pd - (1+\alpha)(R-cd) \geq (x-d)(\gamma s-\alpha c)$ implies that the "composite" equity level $(1+\alpha)R+(\gamma s-\alpha c)x$ is smaller than a threshold level and/or the second-period demand D is stochastically large. On the other hand, when the firm has a high initial inventory level, it intends to pay (partially) the outstanding loan by liquidating part of the on-hand inventory. Specifically, when the salvaging equity level is sufficiently high, i.e., $Q \geq Q_1 \triangleq \frac{(1+\alpha)sF^{-1}(\frac{p-(1+\alpha)^2 s}{p-s})+((1+\alpha)s-p)d}{1+\alpha}$, the firm's optimal policy is to salvage down to the inventory level at which all its outstanding loan is paid off after the first period. In addition, in this scenario, the firm should salvage a positive amount of inventory if $x \geq \frac{pd+(1+\alpha)Q}{(1+\alpha)s}$, i.e., $U \leq -\frac{pd}{1+\alpha}$, which is not related to any initial inventory level. Furthermore, if Q is intermediate, i.e., $Q_2 \leq Q \leq Q_1$ (given that $Q_2 = U_2 + \gamma s x_2 \leq Q_1$), the firm should consider paying off part of the outstanding loan by salvaging-down-to a constant level z^1+d; otherwise when Q further falls below Q_2, the firm should pay back as much as possible to make sure that $U_2 + \gamma s x_2 \geq 0$, i.e., it will not default at the beginning of period 2. From the above discussions, we see that when faced with a negative net cash level, the firm needs to carefully check its current state variables to decide whether to order or salvage, and if so, how much to order or salvage. The following example provides an illustration for these decisions.

Example 1.4.3. *Consider a two-period instance with* $p=3$, $c=1$, $s=0.5$, $\gamma=0.5$, $\alpha=0.5$, $d=1$ *and* D *follows a truncated normal distribution in the interval* $[0,40]$ *with mean 20 and standard deviation 20. Suppose* D *is independent of* d. *The initial inventory level and cash level are* $x=30$ *and* $U=-7$ *in the first period. The optimal salvage-down-to level should be* $\hat{y}^* = \frac{(p-\gamma s)d+(1+\alpha)Q}{(1+\alpha)s-\gamma s} = 29.5$*; when* x *increases to* 32, $\hat{y}^* = z^1+d = 29.8$ *and when* $x=40$, $\hat{y}^* = \frac{pd+(1+\alpha)Q}{(1+\alpha)s} = 30$.

1.5. Numerical Study

In this section, we conduct a variety of numerical experiments to study the structure of the optimal policy under different parametric settings, the impact of the demand uncertainty on the optimal policy, and the percentage profit improvement under inventory-based financing. We test exhaustive combinations of the problem parameters and find the results are quite robust. We fix $T = 2$ in the subsequent experiments. The two-period instances can serve the general purpose of illustrating the major insights carried by inventory-based financing.

1.5.1. Comparative statics

We first study the impact of initial cash level U on the optimal policy. We set the following numerical example parameters as benchmark: $p = 3$, $c = 1$, $s = 0.5$, $\alpha = 0.1$, $\gamma = 0.5$, D_1 and D_2 are truncated normally distributed in the interval $[0, 10]$ with $\mu_1 = \mu_2 = 5$ and $\sigma_1 = \sigma_2 = 5$, and D_1 is independent of D_2. Figure 3 plots the optimal policy under different cash levels, namely, $U = 2, 0, -2$.

We observe from Figure 3 that given any initial inventory level x, the optimal inventory level y^* is larger for a higher initial cash level U. This concurs with our intuition that the firm will order more if it is endowed with a larger amount of cash. In addition, the shape of the optimal policy is similar as those presented in Examples 1 and 2. When the inventory level x is small, the firm optimally over-orders by inventory-based financing for $U = 2$ and $U = 0$ while salvages to pay back some outstanding loan for $U = -2$; as x increases, the line for y^* under either $U = 2$ and $U = 0$ becomes flat, meaning the firm should

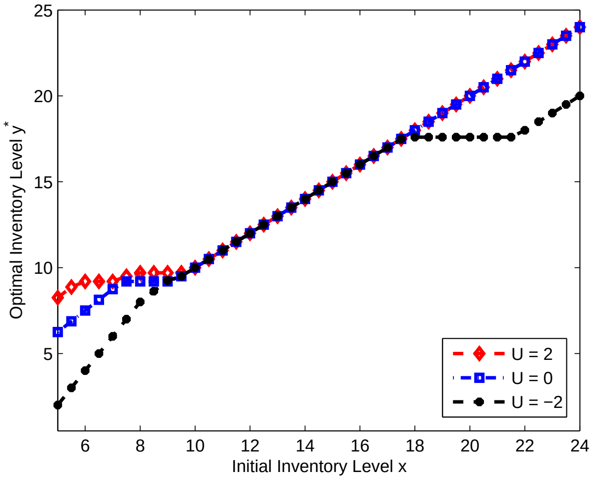

Figure 1.3: Optimal Inventory Level for Different x with $U = 2, 0, -2$

order up to a constant level; for larger x, the three lines for optimal policy begin to overlap representing the firm will neither order nor salvage; for even larger x, the firm will still keep the inventory at the initial level x for $U = 2$ and $U = 0$ but salvages some inventory for $U = -2$.

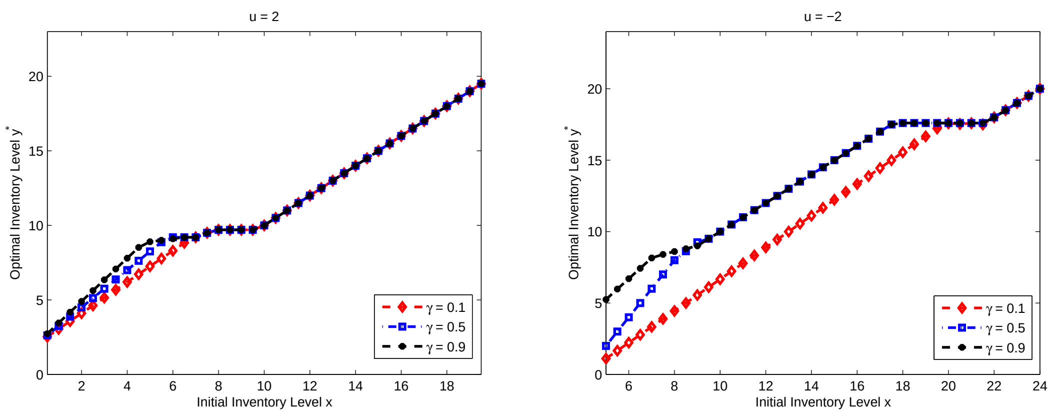

Figure 1.4: Optimal Inventory Level for Different x with $\gamma = 0.1, 0.5, 0.9$

Figure 1.4 shows the results for different loan-to-value ratio γ and for both $U = 2$ and $U = -2$. We find the firm will use more inventory-based financing for a given x and U with a larger γ. This is straightforward since with a larger γ,

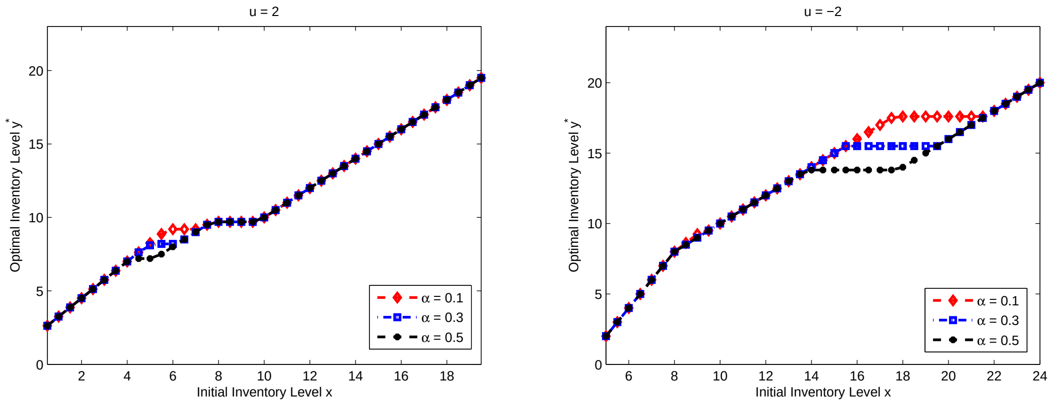

Figure 1.5: Optimal Inventory Level for Different x with $\alpha = 0.1,\ 0.3,\ 0.5$

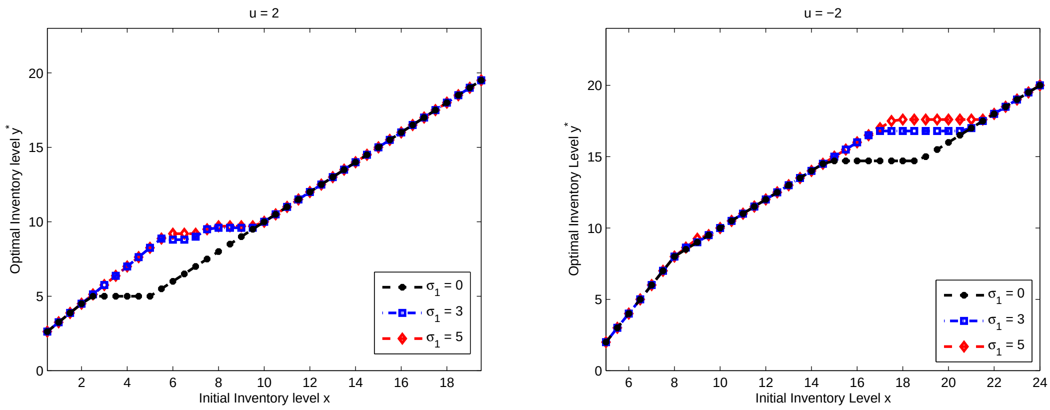

Figure 1.6: Optimal Inventory Level for Different x with $\sigma_1 = 0,\ 3,\ 5$

the firm can afford more inventory when using the financing scheme. Figure 1.5 shows that for different loan interest rate α, the optimal y^* is getting smaller as α grows. This is due to the fact that the higher the borrowing cost, the less the firm will order under the inventory-based financing scheme. Figure 1.6 investigates the influence of the standard deviation σ_1 in the first period. We find that the larger σ_1, the more inventory the firm will order. This reflects the simple fact that the firm has to order more to cope with higher demand uncertainty. Particularly, if the demand is deterministic in the first period, it is optimal to fulfill that amount and possibly carry some additional inventory to prepare for random future demand. In addition, Figures 4, 5, and 6 share a common characteristic that the line of the optimal policy is more drastically differentiated from each other under $U = -2$ than under $U = 2$. This shows that the change of the problem parameters will exert more significant effect on the optimal policy for "poor" firms that are short of cash.

1.5.2. Profit Improvement Percentage

We define the profit improvement percentage of the optimal policy as the percentage of increase in the expected profit compared with the optimal expected profit under the policy without inventory-based financing, i.e.,

$$PI = (\frac{V_{IF}}{V_{NIF}} - 1) * 100\%$$

Table 1.1: Profit Improvement Percentage under Demand Pattern A

α, γ, x	$\mu_1 = 3$		$\mu_1 = 5$		$\mu_1 = 7$	
	$U = 1$	$U = -1$	$U = 1$	$U = -1$	$U = 1$	$U = -1$
0.1, 0.3, 3	2.44 %	20.8 %	2.76 %	21.15 %	3.06 %	21.44 %
0.1, 0.3, 5	1.47 %	12.84 %	1.94 %	13.6 %	2.42 %	14.33 %
0.1, 0.5, 3	3.9 %	50.66 %	4.44 %	51.82 %	4.96 %	52.77 %
0.1, 0.5, 5	2.17 %	21.34 %	2.92 %	23.01 %	3.7 %	24.65 %
0.3, 0.3, 3	1.81 %	19.85 %	2.18 %	20.23 %	2.52 %	20.54 %
0.3, 0.3, 5	0.75 %	11.88 %	1.25 %	12.72 %	1.76 %	13.5 %
0.3, 0.5, 3	2.85 %	48.41 %	3.48 %	49.67 %	4.07 %	50.69 %
0.3, 0.5, 5	0.97 %	19.39 %	1.78 %	21.21 %	2.61 %	22.95 %

To report the numerical study, we investigate two demand patterns A and B: pattern A means we fix the second-period demand mean $\mu_2 = 5$ and vary the first-period demand mean μ_1 from low ($\mu_1 = 3$), to medium ($\mu_1 = 5$), and then to high ($\mu_1 = 7$) whereas pattern B stands for the opposite, in which $\mu_1 = 5$ and μ_2 is varied from low to high. Within this two demand patterns, we group the instances as follows: we construct 16 scenarios by letting $U = 1$ and $U = -1$, $x = 3$ and $x = 5$, $\alpha = 0.1$ and $\alpha = 0.3$ and, $\gamma = 0.3$ and $\gamma = 0.5$. Note we separate the cash level U from α, γ and x in Table 1 and 2 since the performance under a positive cash level is much different from that under a negative cash level. First, from Table 1, we have the following observations for pattern A:

(1) The percentage profit improvement is more significant under lower interest rate ($\alpha = 0.1$), higher loan-to-value ratio ($\gamma = 0.5$) and lower initial inventory level ($x = 3$).

(2) For both cash levels under consideration, the profit improvement is higher with a larger μ_1, i.e., $PI(\mu_1 = 7) > PI(\mu_1 = 5) > PI(\mu_1 = 3)$. This is because anticipating a larger demand for the current period, a firm short of cash and inventory will over-order, which can somehow satisfy more demand, thereby contributing more to the firm's surplus.

(3) The relative profit improvement is higher under the negative cash level $U = -1$ than under the positive cash level $U = 1$. This is straightforward since the financing scheme becomes more valuable to a firm carried with outstanding loan.

Next, we present main observations for pattern B. We find that the above statements (1) and (3) hold in this pattern but some assertion in (2) does not. In particular, except for the case that $x = 3$, $U = -1$ where it still follows $PI(\mu_2 = 7) > PI(\mu_2 = 5) > PI(\mu_2 = 3)$, we observe the opposite relationship $PI(\mu_2 = 3) > PI(\mu_2 = 5) > PI(\mu_2 = 7)$ for all other cases. This is because the inventory-based financing is most beneficial when $x = 3$, $U = -1$, at which the scheme can still eventually increase the relative profit improvement given a pair of increasing benchmark profits, i.e., from $V_{NIF}(\mu_2 = 3)$ to $V_{NIF}(\mu_2 = 7)$. For other cases, although the scheme can always increase the expected profit for any demand mean μ_2, the relative profit increase actually drops due to the increasing benchmark profits. Furthermore, compared with Pattern A, the variation of p becomes narrower in B which means the relative profit improvement gets comparatively smaller. We interpret this as follows: Pattern A concentrates on the demand pattern variation for the first period while Pattern B focuses on the second period. Clearly, the profit is more sensitive to the first-period demand variation than to the second-period variation, leading to different magnitudes of profit improvements between the two demand patterns.

Table 1.2: Profit Improvement Percentage under Demand Pattern B

α, γ, x	$\mu_2 = 3$		$\mu_2 = 5$		$\mu_2 = 7$	
	$U = 1$	$U = -1$	$U = 1$	$U = -1$	$U = 1$	$U = -1$
0.1, 0.3, 3	2.88 %	19.96 %	2.76 %	21.15 %	2.66 %	22.21 %
0.1, 0.3, 5	2.04 %	14.43 %	1.94 %	13.6 %	1.87 %	12.9 %
0.1, 0.5, 3	4.63 %	48.11 %	4.44 %	51.82 %	4.29 %	55.3 %
0.1, 0.5, 5	3.07 %	24.42 %	2.92 %	23.01 %	2.8 %	21.82 %
0.3, 0.3, 3	2.29 %	19.1 %	2.18 %	20.23 %	2.09 %	21.24 %
0.3, 0.3, 5	1.32 %	13.52 %	1.25 %	12.72 %	1.19 %	12.04 %
0.3, 0.5, 3	3.65 %	46.13 %	3.48 %	49.67 %	3.34 %	52.97 %
0.3, 0.5, 5	1.87 %	22.56 %	1.78 %	21.21 %	1.69 %	20.06 %

1.6. Concluding Remarks

In this paper, we study a stochastic dynamic inventory control problem for a capital-constrained firm which can borrow short-term loans to support its production under inventory-based financing scheme. In our model, the loan capacity is closely related to the firm's equity level, which guarantees a risk-free position to the bank. We partially derive the optimal inventory policy and characterize the interdependence of its inventory flow and financial flow. We show an purchasing- or salvaging-equity-level-dependent base-stock policy is optimal in each period. More specifically, when the firm's purchasing or salvaging equity level is sufficiently high, it should either order or salvage to a certain constant level whereas when it has a relatively low equity level, it should obtain some loans by inventory-based financing to secure additional inventory.

We also find some underlying managerial insights through extensive numerical experiments. Ceteris paribus, under lower interest rate, higher loan-to-value ratio, negative initial cash level and lower initial inventory level, the firm intends to order more and the percentage profit improvement becomes more significant compared with the optimal policy in which there is no inventory-based financing. Furthermore, the demand pattern also plays an important role on the profitability of the financing scheme. The improvement is most significant under the case where the current or future demand is stochastically high, and this impact is more sensitive to the current demand than to the future demand.

Our models have some limitations. For example, we assume there is no holding cost and no deposit rate in our formation. In some business situations, these two factors may not be negligible. Another interesting direction is to consider varied salvage values for the collateral. For instance, the bank should adjust the loan rate or the loan-to-value ratio dynamically to accommodate a fluctuated collateral price in the spot or future market. Relaxing these assumptions could possibly lead to new insights, which, however, will also impose significant technical challenges that are beyond the current scope of this article.

1.7. Appendix

Proof of Proposition 1.3.1

Proof. We prove by induction. It is obvious that $\hat{V}_{T+1}(x_{T+1}, \hat{U}_{T+1}, \hat{l}_{T+1})$ is jointly concave in $(x_{T+1}, \hat{U}_{T+1}, \hat{l}_{T+1})$ since all terms are linear in $(x_{T+1}, \hat{U}_{T+1}, \hat{l}_{T+1})$. Suppose $\hat{V}_{t+1}(x_{t+1}, \hat{U}_{t+1}, \hat{l}_{t+1})$ is jointly concave in $(x_{t+1}, \hat{U}_{t+1}, \hat{l}_{t+1})$ for $1 \leq t \leq T$. We shall prove $\hat{V}_t(x_t, \hat{U}_t, \hat{l}_t)$ is jointly concave in $(x_t, \hat{U}_t, \hat{l}_t)$. To do so, we shall first show that

$$\hat{V}_{t+1}((x_t - z_t + \hat{q}_t - d_t)^+, (\hat{U}_t - \alpha\hat{l}_t + sz_t + b_t - c\hat{q}_t) + p\min\{d_t, x_t - z_t + \hat{q}_t\}, \hat{l}_t + b_t)$$

is jointly concave in $(x_t, \hat{U}_t, \hat{l}_t, z_t, \hat{q}_t, b_t)$. Namely, we shall show that for any $(x_t^1, \hat{U}_t^1, \hat{l}_t^1, z_t^1, \hat{q}_t^1, b_t^1)$, $(x_t^2, \hat{U}_t^2, \hat{l}_t^2, z_t^2, \hat{q}_t^2, b_t^2) \in S_t$ and $0 \leq \eta \leq 1$, we have

$$\begin{aligned}
& \hat{V}_{t+1}((\eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2) - d_t)^+, \\
& (\eta(\hat{U}_t^1 - \alpha\hat{l}_t^1 + sz_t^1 + b_t^1 - c\hat{q}_t^1) + (1-\eta)(\hat{U}_t^2 - \alpha\hat{l}_t^2 + sz_t^2 + b_t^2 - c\hat{q}_t^2)) \\
& +p\min\{d_t, \eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2)\}, \eta(\hat{l}_t^1 + b_t^1) + (1-\eta)(\hat{l}_t^1 + b_t^1)) \\
\geq\ & \eta\hat{V}_{t+1}((x_t^1 - z_t^1 + \hat{q}_t^1 - d_t)^+, (\hat{U}_t^1 - \alpha\hat{l}_t^1 + sz_t^1 + b_t^1 - c\hat{q}_t^1) \\
& +p\min\{d_t, x_t^1 - z_t^1 + \hat{q}_t^1\}, \hat{l}_t^1 + b_t^1) \\
& +(1-\eta)\hat{V}_{t+1}((x_t^2 - z_t^2 + \hat{q}_t^2 - d_t)^+, (\hat{U}_t^2 - \alpha\hat{l}_t^2 + sz_t^2 + b_t^2 - c\hat{q}_t^2) \\
& +p\min\{d_t, x_t^2 - z_t^2 + \hat{q}_t^2\}, \hat{l}_t^2 + b_t^2).
\end{aligned}$$

Let

$$\begin{aligned}\Delta &= -\left(\eta\left(x_t^1 - z_t^1 + \hat{q}_t^1\right) + (1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2\right) - d_t\right)^+ \\ &\quad + \left(\eta\left(x_t^1 - z_t^1 + \hat{q}_t^1 - d_t\right)^+ + (1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2 - d_t\right)^+\right) \geq 0.\end{aligned}$$

Note that

$$\begin{aligned}&(\eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2) - d_t)^+ + \Delta \\ =\ & \eta\left(x_t^1 - z_t^1 + \hat{q}_t^1 - d_t\right)^+ + (1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2 - d_t\right)^+\end{aligned}$$

and

$$\begin{aligned}&p\min\{d_t, \eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2)\} - p\Delta \\ =\ & p\min\{d_t, \eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2)\} \\ & +p\left(\eta\left(x_t^1 - z_t^1 + \hat{q}_t^1\right) + (1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2\right) - d_t\right)^+ \\ & -p\left(\eta\left(x_t^1 - z_t^1 + \hat{q}_t^1 - d_t\right)^+ + (1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2 - d_t\right)^+\right) \\ =\ & p\min\{d_t, \eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2)\} \\ & +p\left(\eta\left(x_t^1 - z_t^1 + \hat{q}_t^1\right) + (1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2\right)\right) \\ & -p\min\{\eta\left(x_t^1 - z_t^1 + \hat{q}_t^1\right) + (1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2\right), d_t\} \\ & -p(\eta\left(x_t^1 - z_t^1 + \hat{q}_t^1\right) - \eta\min\{x_t^1 - z_t^1 + \hat{q}_t^1, d_t\} \\ & +(1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2\right) - (1-\eta)\min\{x_t^2 - z_t^2 + \hat{q}_t, d_t\}) \\ =\ & p\eta\min\{x_t^1 - z_t^1 + \hat{q}_t^1, d_t\} + p(1-\eta)\min\{x_t^2 - z_t^2 + \hat{q}_t, d_t\}.\end{aligned}$$

Hence

$$
\begin{aligned}
&\hat{V}_{t+1}((\eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2) - d_t)^+, \\
&\qquad (\eta(\hat{U}_t^1 - \alpha\hat{l}_t^1 + sz_t^1 + b_t^1 - c\hat{q}_t^1) + (1-\eta)(\hat{U}_t^2 - \alpha\hat{l}_t^2 + sz_t^2 + b_t^2 - c\hat{q}_t^2)) \\
&\qquad\qquad + p\min\{d_t, \eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2)\}, \\
&\qquad\qquad\qquad \eta(\hat{l}_t^1 + b_t^1) + (1-\eta)(\hat{l}_t^2 + b_t^2)) \\
\geq\ &\hat{V}_{t+1}((\eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2) - d_t)^+ + \Delta, \\
&\qquad (\eta(\hat{U}_t^1 - \alpha\hat{l}_t^1 + sz_t^1 + b_t^1 - c\hat{q}_t^1) + (1-\eta)(\hat{U}_t^2 - \alpha\hat{l}_t^2 + sz_t^2 + b_t^2 - c\hat{q}_t^2)) \\
&\qquad\qquad + p\min\{d_t, \eta(x_t^1 - z_t^1 + \hat{q}_t^1) + (1-\eta)(x_t^2 - z_t^2 + \hat{q}_t^2) - \Delta\}, \\
&\qquad\qquad\qquad \eta(\hat{l}_t^1 + b_t^1) + (1-\eta)(\hat{l}_t^2 + b_t^2)) \\
=\ &\hat{V}_{t+1}(\eta\left(x_t^1 - z_t^1 + \hat{q}_t^1 - d_t\right)^+ + (1-\eta)\left(x_t^2 - z_t^2 + \hat{q}_t^2 - d_t\right)^+, \\
&\qquad (\eta(\hat{U}_t^1 - \alpha\hat{l}_t^1 + sz_t^1 + b_t^1 - c\hat{q}_t^1) + (1-\eta)(\hat{U}_t^2 - \alpha\hat{l}_t^2 + sz_t^2 + b_t^2 - c\hat{q}_t^2)) \\
&\qquad\qquad + p\eta\min\{x_t^1 - z_t^1 + \hat{q}_t^1, d_t\} + p(1-\eta)\min\{x_t^2 - z_t^2 + \hat{q}_t, d_t\}\}, \\
&\qquad\qquad\qquad \eta(\hat{l}_t^1 + b_t^1) + (1-\eta)(\hat{l}_t^2 + b_t^2)) \\
\geq\ &\eta\hat{V}_{t+1}((x_t^1 - z_t^1 + \hat{q}_t^1 - d_t)^+, (\hat{U}_t^1 - \alpha\hat{l}_t^1 + sz_t^1 + b_t^1 - c\hat{q}_t^1) \\
&\qquad\qquad + p\min\{d_t, x_t^1 - z_t^1 + \hat{q}_t^1\}, \hat{l}_t^1 + b_t^1) \\
&+ (1-\eta)\hat{V}_{t+1}((x_t^2 - z_t^2 + \hat{q}_t^2 - d_t)^+, (\hat{U}_t^2 - \alpha\hat{l}_t^2 + sz_t^2 + b_t^2 - c\hat{q}_t^2) \\
&\qquad\qquad + p\min\{d_t, x_t^2 - z_t^2 + \hat{q}_t^2\}, \hat{l}_t^2 + b_t^2).
\end{aligned}
$$

where the first inequality is by Lemma 1.3.2 and the second inequality is by inductive assumption. So we have shown that

$$
\begin{aligned}
&\hat{V}_{t+1}((x_t - z_t + \hat{q}_t - d_t)^+, (\hat{U}_t - \alpha\hat{l}_t + sz_t + b_t - c\hat{q}_t) \\
&\qquad\qquad + p\min\{d_t, x_t - z_t + \hat{q}_t\}, \hat{l}_t + b_t)
\end{aligned}
$$

is jointly concave in $(x_t, \hat{U}_t, \hat{l}_t, z_t, \hat{q}_t, b_t)$. Since

$$\begin{aligned}\hat{V}_t(x_t, \hat{U}_t, \hat{l}_t) &= \max_{\substack{(z_t, \hat{q}_t, b_t) \text{ satisfies} \\ \text{inequalities (1.1) -- (1.5)}}} \mathsf{E}\Big[\hat{V}_{t+1}((x_t - z_t + \hat{q}_t - d_t)^+, \\ &\qquad (\hat{U}_t - \alpha\hat{l}_t + sz_t + b_t - c\hat{q}_t) + p\min\{d_t, x_t - z_t + \hat{q}_t\}, \hat{l}_t + b_t)\Big]\end{aligned}$$

and $\hat{S}_t \triangleq \{(x_t, \hat{U}_t, l_t, z_t, \hat{q}_t, b_t) : (x_t, \hat{U}_t, \hat{l}_t, z_t, \hat{q}_t, b_t)$ satisfies inequalities (1.1) - (1.5)$\}$ is a convex set, we conclude that $\hat{V}_t(x_t, \hat{U}_t, \hat{l}_t)$ is jointly concave in $(x_t, \hat{U}_t, \hat{l}_t)$. □

Proof of Proposition 1.4.1

Proof. First we show when $U_t \geq 0$, the value function $V_t(x_t, U_t) = G_t^d(\max\{a_t^d, x_t\}) + H_t(R_t)$, where

$$H_t(R_t) = \begin{cases} R_t, & R_t \geq ca_t^d; \\ \Pi_t^d(R_t/c, R_t) - G_t^d(a_t^d), & ca_t^{br} \leq R_t \leq ca_t^d; \\ \Pi_t^{br}(y_t^*, R_t) - G_t^d(a_t^d), & 0 \leq R_t \leq ca_t^{br}. \end{cases}$$

Suppose for period $t+1$ it follows $V_{t+1}(x_{t+1}, U_{t+1}) = G_{t+1}^d(\max\{a_{t+1}^d, x_{t+1}\}) + R_{t+1}$ if $R_{t+1} \geq ca_{t+1}$ and $U_{t+1} \geq 0$. In period t, assume $R_t \geq ca_t^d$ and $y_t \leq R_t/c$, we have $U_{t+1} = p\min\{y_t, D_t\} + (R_t - cy_t) \geq 0$ and $R_{t+1} = (p-c)\min\{y_t, D_t\} + R_t \geq ca_t^d \geq ca_{t+1}^d$. Therefore by the inductive assumption for V_{t+1}, it holds,

$$\begin{aligned}\Pi_t^d(y_t, R_t) &= \mathsf{E}\Big[V_{t+1}((y_t - D_t)^+, p\min\{D_t, y_t\} + (R_t - cy_t))\Big] \\ &= \mathsf{E}\Big[((p-c)\min\{y_t, D_t\} + R_t + G_{t+1}^d(\max\{a_{t+1}^d, (y_t - D_t)^+\})\Big] \\ &= R_t + G_t^d(y_t)\end{aligned}$$

Notice a_t^d is the maximizer of $G_t^d(y_t)$, we have $\frac{dG_t^d(x_t)}{dy_t} \leq 0$ if $x_t \geq a_t^d$, which implies that the firm should never order if her inventory level is above x_t. On the other hand, if $x_t < a_t^d$ and since $R_t \geq ca_t^d$ holds, it is optimal for the firm to order up to the constant inventory level a_t^d. Thus it follows $V_t(x_t, U_t) = G_t^d(\max\{a_t^d, x_t\}) + R_t$ when $R_t \geq ca_t^d$.

Next we should prove when $ca_t^{br} \leq R_t \leq ca_t^d$, it follows $y_t^* = R_t/c$. To prove this, it suffices to prove that $cy_t^{br} \leq R_t \leq cy_t^d$ in this case. If $R_t \geq ca_{t+1}^d$, the result $R_t/c \leq y_t^d$ is clearly true; otherwise if $ca_t^{br} \leq R_t \leq ca_{t+1}^d$ and when $y_t \leq R_t/c$, by the inductive assumption for $t+1$,

$$\Pi_t^d(y_t, R_t) = G_{t+1}^d(a_{t+1}^d) + \mathsf{E}\Big[H_{t+1}((p-c)\min\{D_t, y_t\} + R_t)\Big]$$

Then taking the partial derivative with respect to y_t and letting $y_t = R_t/c$,

$$\begin{aligned} \frac{\partial \Pi_t^d(y_t, R_t)}{dy_t}\Big|_{y_t = R_t/c} &= \mathsf{E}\Big[((p-c)1_{\{R_t/c \leq D_t\}})H'_{t+1}((p-c)\min\{D_t, R_t/c\} + R_t)\Big] \\ &\geq 0 \end{aligned}$$

where $1_{\{\ \}}$ is the indicator function and the inequality holds because $H_t(R_t)$ increases in R_t. Thus we conclude $R_t/c \leq y_t^d$.

Then we prove $R_t/c \geq y_t^{br}$. If $R_t \geq ca_{t+1}^d$, according to the definition of $\Pi_t^d(y_t, R_t)$, we can easily show $\frac{\partial \Pi_t^{br}(y_t, R_t)}{dy_t}|_{y_t = R_t/c} \leq 0$; on the other hand if $ca_t^{br} \leq R_t \leq ca_{t+1}^d$, according to Lemma 5, it follows $a_t^{br} = F_t^{-1}(\frac{p-(1+\alpha)c}{p-c})$. When $y_t \leq R_t/c$, by the induction for $t+1$,

$$\begin{aligned} \Pi_t^{br}(y_t, R_t) &= \mathsf{E}\Big[V_{t+1}((y_t - D_t)^+, p\min\{D_t, y_t\} + (1+\alpha)(R_t - cy_t))\Big] \\ &= \mathsf{E}\Big[G_{t+1}^d(a_{t+1}^d) + H_{t+1}((p-c)\min\{y_t, D_t\} + (1+\alpha)R_t - \alpha c y_t)\Big] \end{aligned}$$

Taking the partial derivative with respect to y_t and letting $y_t = R_t/c$, we have

$$\begin{aligned}
& \frac{\partial \Pi_t^{br}(y_t, R_t)}{dy_t}|_{y_t=R_t/c} \\
= \;& \mathsf{E}\Big[((p-c)1_{\{R_t/c\le D_t\}} - \alpha c)H'_{t+1}((p-c)\min\{D_t, R_t/c\} + R_t)\Big] \\
= \;& (p-c)\bar{F}_t(R_t/c)H'_{t+1}(pR_t/c) - \alpha c\mathsf{E}\Big[H'_{t+1}((p-c)\min\{D_t, R_t/c\} + R_t)\Big] \\
\le \;& ((p-c)\bar{F}_t(R_t/c) - \alpha c)\mathsf{E}\Big[H'_{t+1}((p-c)\min\{D_t, R_t/c\} + R_t)\Big] \\
\le \;& 0
\end{aligned}$$

The first inequality is because $H'_{t+1}(\cdot)$ is decreasing and the second inequality is because $R_t \ge ca_t^{br}$ and $a_t^{br} = F_t^{-1}(\frac{p-(1+\alpha)c}{p-c})$. Hence, $R_t/c \ge a_t^{br}$ and we have proved $H_t(R_t) = \Pi_t^d(R_t/c, R_t) - G_t^d(a_t^d)$ when $ca_t^{br} \le R_t \le ca_t^d$.

Finally, when $R_t \le ca_t^{br}$, it suffices to prove $\Pi_t(y_t, R_t)$ increases in y_t when $y_t \le R_t/c$. If $R_t \ge ca_{t+1}^d$, it is easy to verify that $\hat{y}_t^* = a_t^{br}$; otherwise if $0 \le R_t \le ca_{t+1}^d$, when $R_{t+1} \ge cx_{t+1}$, according to the induction for period $t+1$ and let $\hat{y}_t \le R_t/c$, we have

$$\Pi_t(y_t, R_t) = G_{t+1}^d(a_{t+1}^d) + \mathsf{E}\Big[H_{t+1}((p-c)\min\{D_t, y_t\} + R_t)\Big].$$

Note that $H_{t+1}(\cdot)$ is increasing, then we can conclude that $\Pi_t(y_t, y_t)$ is increasing in y_t when $y_t \le R_t/c$. Therefore $\hat{y}_t^* \ge R_t/c$ and it shows $H_t(R_t) = \Pi_t^{br}(y_t^*, R_t) - G_t^d(a_t^d)$ when $0 \le R_t \le ca_t^{br}$. □

Proof of Proposition 1.4.2

Proof. We claim that in period t, the firm is not optimal to salvage down to the inventory level below Q_t/s. This is because when it pays back all the outstanding loan U_t by salvaging $-U_t/s$ units, she has no incentive to further salvage since the cash level is already positive at this point. In this sense, we would prove that given $Q_t \geq sa_t^s$, the optimal salvage-down-to level $y_t^s \leq Q_t/s$. The reason is when $Q_t \geq sa_t^s$ and let $y_t^s = Q_t/s$, we can decompose the value function $\Pi_t^s(y_t, Q_t)$ as follows,

$$
\begin{aligned}
& \Pi_t^s(y_t, Q_t) \\
= \quad & \mathsf{E}\Big[V_{t+1}((y_t - D_t)^+, p\min\{y_t, D_t\} + (1+\alpha)(Q_t - sy_t))\Big] \\
= \quad & \mathsf{E}\Big[p\min\{y_t, D_t\} + (1+\alpha)(Q_t - sy_t) + c(y_t - D_t)^+ + G_{t+1}^d(\max\{a_{t+1}^d, (y_t - D_t)^+\})\Big] \\
= \quad & (1+\alpha)Q_t + G_t^s(y_t)
\end{aligned}
$$

Note a_t^s is the maximizer of $G_t^s(y_t)$, the above equation implies $\frac{d\Pi_t^s(Q_t/s, Q_t)}{dy_t} \leq 0$. As mentioned earlier that the optimal salvaged-down-to level should be no less than Q_t/s, we conclude $y_t^s = Q_t/s$ in this scenario. □

Proof of Proposition 1.4.3

Proof. We define $z = y - d \leq (R + \gamma sx)/c - d$, which represents the remaining inventory level after meeting the current demand. We consider two cases.

Case (a): $x - d \leq z \leq \frac{R+\gamma sx}{c} - d$, i.e., $x \leq y \leq \frac{R+\gamma sx}{c}$, meaning the firm orders inventory at the beginning of period 1. At this point, the optimal equation becomes

$$V_1(x,U) = \max_{x-d\le z\le \frac{R+\gamma sx}{c}-d} V_2\big(z, pd+(1+\alpha)(R-cz-cd)\big).$$

(a.1) $U_2 \ge 0$. It requires $z \le z_1 \triangleq \frac{pd+(1+\alpha)R}{(1+\alpha)c} - d$. Note $x_2 = z$ and $R_2 = pd+(1+\alpha)(R-cd)-\alpha cz$ and suppose for some z, we have $-(1+\alpha)c+s+(p-s)\bar{F}(\frac{R_2+\gamma sx_2}{c}) \le 0$ which implies at period 2 the firm's optimal policy is to order products by using his on-hand cash or by collateralizing part of (not all) the available inventory. Specifically, when $s-c+(p-s)\bar{F}(x_2) \le 0$ we have $y_2^* = x_2$ and $V_2(x_2,U_2) = \mathsf{E}[R_2+(s-c)x_2+(p-s)\min\{x_2,D\}]$. By calculation, it holds $\frac{dV_2(z,pd+(1+\alpha)(R-cz-cd))}{dz} = (p-s)\bar{F}(x_2)-(1+\alpha)c+s \le 0$ since we have the range condition $s-c+(p-s)\bar{F}(x_2) \le 0$. This means a smallest z should be optimal within this range; when $s-c+(p-s)\bar{F}(x_2) \ge 0$ and $s-c+(p-s)\bar{F}(\frac{R_2}{c}) \le 0$ we have $y_2^* = a_2^d$ and $V_2(x_2,U_2) = \mathsf{E}[R_2+(s-c)a_2^d+(p-s)\min\{a_2^d,D\}]$; when $s-c+(p-s)\bar{F}(x_2) \le 0$ and $s-c+(p-s)\bar{F}(\frac{R_2}{c}) \le 0$ we have $y_2^* = R_2/c$ and $V_2(x_2,U_2) = \mathsf{E}[\frac{sR_2}{c}+(p-s)\min\{R_2/c,D\}]$; when $s-c+(p-s)\bar{F}(x_2) \ge 0$ and $s-(1+\alpha)c+(p-s)\bar{F}(\frac{R_2+\gamma sx_2}{c}) \le 0$ we have $y_2^* = a_2^{br}$ and $V_2(x_2,U_2) = \mathsf{E}[(1+\alpha)R_2+(s-c-\alpha c)a_2^{br}+(p-s)\min\{a_2^{br},D\}]$. Note that for all these z in different range, the optimal value V_2 increases in the equity level $R_2 = pd+(1+\alpha)(R-cd)-\alpha cz$ which however is a decreasing function of z. Therefore we conclude that the optimal z^* should take the smallest value within the range $-(1+\alpha)c+s+(p-s)\bar{F}(\frac{R_2+\gamma sx_2}{c}) \le 0$.

On the other hand, for other $z \in [x-d, \frac{R+\gamma sx}{c}-d]$, it should be $-(1+\alpha)c+s+(p-s)\bar{F}(\frac{R_2+\gamma sx_2}{c}) \ge 0$ which indicates that the firm's optimal policy is to order by using up all the equality under the inventory-based financing. At this point, we have $y_2^* = \frac{R_2+\gamma sx_2}{c}$ and $V_2(x_2,U_2) = \mathsf{E}[(1+\alpha)R_2+(s-c-\alpha c)\frac{R_2+\gamma sx_2}{c}+$

$(p-s)\min\{\frac{R_2+\gamma s x_2}{c}, D\}]$. By calculation it follows,

$$\begin{aligned}
&\frac{dV_2(z, pd+(1+\alpha)(R-cd-cz))}{dz} \\
= &\left(\frac{\partial V_2}{\partial x} - (1+\alpha)c\frac{\partial V_2}{\partial u}\right)(z, pd+(1+\alpha)(R-cd-cz)) \\
= &\frac{p-s}{c}(\gamma s-\alpha c)\bar{F}\left(\frac{pd+(1+\alpha)(R-cd)+(\gamma s-\alpha c)z}{c}\right) - (\alpha\gamma s+\alpha s+\gamma s-\frac{s}{c}\gamma s)
\end{aligned}$$

For convenience, denote the condition $0 \leq \frac{\alpha\gamma s+\alpha s+\gamma s-\gamma s^2/c}{(p-s)(\gamma s-\alpha c)/c} \leq 1$ as (*). If (*) holds, the interior optimal solution z^0 should satisfy $\bar{F}(\frac{pd+(1+\alpha)(R-cd)+(\gamma s-\alpha c)z^0}{c}) = \frac{\alpha\gamma s+\alpha s+\gamma s-\gamma s^2/c}{(p-s)(\gamma s-\alpha c)/c}$. Also note that $\frac{(1+\alpha)c-s}{p-s} \leq \frac{\alpha\gamma s+\alpha s+\gamma s-\gamma s^2/c}{(p-s)(\gamma s-\alpha c)/c}$, which means for such z^0, it holds $-(1+\alpha)c+s+(p-s)\bar{F}(\frac{R_2+\gamma s x_2}{c}) \geq 0$. Otherwise if (*) breaks down, it is easy to check the above derivative is less then 0 which indicates a smallest z^* within the range.

(a.2) $U_2 \leq 0$ and $U_2+\gamma s x_2 \geq 0$. Then we have $z_1 \leq z \leq z_2 \triangleq \frac{pd+(1+\alpha)(R-cd)}{(1+\alpha)c-\gamma s}$. Suppose for some z, it holds that $-\alpha s+(p-s)\bar{F}(\frac{Q_2}{s}) \leq 0$ where $Q_2 = pd+(1+\alpha)(R-cd)+(s-\alpha c-c)z$, which means at period 2 the firm's optimal policy is to pay back all the loan outside by salvaging some on-hand inventory. At this point, we have $y_2^* = Q_2/s$ and $V_2(x_2, U_2) = \mathsf{E}[Q_2+(p-s)\min\{Q_2/s, D\}]$ implying that the optimal Q_2 should be as large as possible, leading to z^* as small as possible in this range; if $-\alpha s+(p-s)\bar{F}(\frac{Q_2}{s}) \geq 0$ and $-\alpha s+(p-s)\bar{F}(x_2) \leq 0$ holds, meaning that at period 2 the firm's optimal policy is to salvage down to a certain level a_2^{sr}. At this point, $y_2^* = a_2^{sr}$ and $V_2(x_2, U_2) = \mathsf{E}[(1+\alpha)Q_2 - \alpha s a_2^{sr} + (p-s)\min\{a_2^{sr}, D\}]$, which also implies a smallest z^*; if $-\alpha s+(p-s)\bar{F}(x_2) \geq 0$ and $-(1+\alpha)c+s+(p-s)\bar{F}(x_2) \leq 0$ follows, meaning that at period 2 the firm's optimal policy is to keep the inventory level at $y_2^* = x_2$. At this point, $y_2^* = x_2$ and $V_2(x_2, U_2) = \mathsf{E}[(1+\alpha)U_2+s x_2+(p-s)\min\{x_2, D\}]$. By calculation, it holds $\frac{dV_2(z, pd+(1+\alpha)(R-cz-cd))}{dz} = (p-s)\bar{F}(x_2)-(1+\alpha)^2c+s \leq 0$

since $-(1+\alpha)c+s+(p-s)\bar{F}(x_2)\leq 0$, implying a smallest z^* in the range; for the rest of z, we have $-(1+\alpha)c+s+(p-s)\bar{F}(x_2)\geq 0$ and the firm's optimal policy at period 2 is to order additional inventory. As in Case 2, when (*) holds, we might have a non-boundary optimal solution $z^*=z^0$ (note when $\gamma s\geq \alpha c$, $\frac{R+\gamma sx}{c}\leq z_2$); otherwise z^* should take the smallest boundary solution in this range. With these observations we conclude that if condition (*) is satisfied and $y^0\geq y_1$, $y^*=\min\{y^0,\frac{R+\gamma sx}{c}\}$, otherwise we have $y^*=\min\{y_1,\frac{R+\gamma sx}{c}\}$.

(a.3) $U_2+\gamma sx_2\leq 0$. Then we have $z\geq z_2$. According to the requirement of the minimal collateral, the firm has to salvage some inventory to pay back the outstanding loan. After salvaging the minimum quantity, period 2's optimal value function becomes $V_2(\frac{U_2+x_2s}{(1-\gamma)s},\frac{-\gamma sx_2-\gamma U_2}{1-\gamma})$. Since at this time V_2 increases in $Q_2=U_2+sx_2$ while Q_2 is decreasing function of z, z^* should be the smallest one in this range, e.g., $z^*=z_2$. Combine the above three cases, we conclude that if $y^0\geq x$ holds and condition (*) is satisfied, we have $y^*=\min\{y^0,\frac{R+\gamma sx}{c}\}$; otherwise $y^*=x$.

Case (b): $\max\{Q/s-d,0\}\leq z\leq x-d$, i.e., $Q/s\leq y\leq x$. This is the situation in which the firm chooses to salvage. Then the optimal value equation becomes

$$V_1(x,U) \;=\; \max_{\max\{Q/s-d,0\}\leq z\leq x-d} V_2\big(z,(p-s-\alpha s)d+(1+\alpha)Q-(1+\alpha)sz\big)$$

(b.1) $U_2\geq 0$. Then we have $z\leq\frac{(p-s-\alpha s)d+(1+\alpha)Q}{(1+\alpha)s}\triangleq z_3$. Because $U_2\geq 0$, the firm should never salvage inventory in period 2, making the optimal value V_2 only related with $x_2=z$ and $R_2=(p-s-\alpha s)d+(1+\alpha)Q+(c-s-\alpha s)z$. Note $c\geq(1+\alpha)s$, thus z^* should take the largest value in the range, e.g., $z^*=\min\{x-d,z_3\}$ and $y^*=\min\{x,\frac{pd+(1+\alpha)Q}{(1+\alpha)s}\}$.

(b.2) $U_2 \le 0$ and $U_2 + \gamma s x_2 \ge 0$. We have $z_3 \le z \le \frac{(p-s-\alpha s)d+(1+\alpha)Q}{(1+\alpha)s-\gamma s} \triangleq z_4$. Suppose for some z, it holds that $-\alpha s + (p-s)\bar{F}(Q_2/s) \le 0$, e.g., $z \le \frac{(p-s-\alpha s)d+(1+\alpha)Q-sF^{-1}(\frac{p-s-\alpha s}{p-s})}{\alpha s} \triangleq z_5$. At this point, $y_2^* = Q_2/s$ and $V_2(x_2, U_2) = \mathsf{E}[Q_2 + (p-s)\min\{Q_2/s, D\}]$, leading to z^* as small as possible; if $-\alpha s + (p-s)\bar{F}(Q_2/s) \ge 0$ and $-\alpha s + (p-s)\bar{F}(x_2) \le 0$ hold, e.g., $z \ge z_5$ and $z \ge F^{-1}(\frac{p-s-\alpha s}{p-s}) \triangleq z_6$, z^* should also take the smallest value, e.g., $z^* = \max\{z_5, z_6\}$; if for other z, such that $-\alpha s+(p-s)\bar{F}(x_2) \ge 0$ and $-(1+\alpha)c+s+(p-s)\bar{F}(x_2) \le 0$, e.g., $F^{-1}(\frac{p-c-\alpha c}{p-s}) \triangleq z_7 \le z \le z_6$, $y_2^* = x_2$ and $V_2(x_2, U_2) = \mathsf{E}[(1+\alpha)U_2+sx_2+(p-s)\min\{x_2, D\}]$. By calculation, we have $\frac{dV_2(z,(p-s-\alpha s)d+(1+\alpha)Q-(1+\alpha)sz)}{dz} = -(1+\alpha)^2 s + s + (p-s)\bar{F}(x_2)$ and suppose V_2 reaches its local optimal at z^1 in this range, where $z_7 \le z^1 \triangleq F^{-1}(z)\big(\frac{p-(1+\alpha)^2 s)}{p-s}\big) \le z_6$. Then z^* should be z^1 in this range; for other $z \le z_7$, we have $-(1+\alpha)c + s + (p-s)\bar{F}(x_2) \ge 0$. At this point, the optimal value V_2 is related with R_2 and x_2. Note R_2 and x_2 are both increasing in z, leading to the largest z^* in this range, e.g., $z^* = z_7$. With these observations, we conclude:

if $z_5 \ge z_3$, it holds that $z_6 \le z_3 \le z_5$. This leads to $z^* = \min\{x-d, z_3\}$ and $y^* = \min\{x, \frac{pd+(1+\alpha)Q}{(1+\alpha)s}\}$; if $z_5 \le z_3$, (a) $z^* = \min\{x-d, z_3\}$ and $y^* = \min\{x, \frac{pd+(1+\alpha)Q}{(1+\alpha)s}\}$ if $z^1 \le z_3$; (b) $z^* = \min\{x-d, z^1\}$ and $y^* = \min\{x, z^1+d\}$ if $z_3 \le z^1 \le z_4$; (c) $z^* = \min\{x-d, z_4\}$ and $y^* = \min\{x, \frac{(p-\gamma s)d+(1+\alpha)Q}{(1+\alpha)s-\gamma s}\}$ if $z^1 \ge z_4$.

(b.3) $U_2 + \gamma s x_2 \le 0$. We have $z \ge \frac{(p-s-\alpha s)d+(1+\alpha)Q}{(1+\alpha)s-\gamma s}$. In this case it is obvious that $z^* = \frac{(p-s-\alpha s)d+(1+\alpha)Q}{(1+\alpha)s-\gamma s}$ in this range.

Finally, combining (a) and (b) together, and note that given x and U, the value function $V_2\big((y-d)^+, p\min\{y,d\} + (U + s(x-y)^+ - c(y-x)^+)^+ - (1+\alpha)(c(y-x)^+ - U - s(x-y)^+)^+\big)$ is concave in y within the constraint range, we can verify that the stated property holds. □

CHAPTER 2

When Quick Response Meets Trade Credit

2.1. Introduction

Quick response (QR) strategy has become increasingly popular and proved to be effective in solving supply chain mismatches. Originally, QR was developed as an inventory management strategy with the idea to respond quickly to market changes by cutting the lead time (Iyer and Bergen, 1997). Nowadays, with the advance in many new technologies such as RFID systems and big data analysis, QR system is widely employed in many kinds of industries over the world, from fast fashion industry to consumer electronics (Choi and Sethi, 2010). Generally speaking, the essence of this strategy is the short production and distribution lead time which enables the firm to take advantage of the market updates. This kind of benefit has been deeply studied by numerous researchers and in practice it helps firms adjust their inventory levels and thus to better match the supply with uncertain demand. On the other hand, capital constraint or cash turnover may play a vital role when such a firm implementing the quick response techniques. In view of this, some firms make the agreement with their upstream partners for the permission of trade credit financing, which refers to the credit extended to the firm for the purchase of products. In the fast fashion industry, some local Spanish, Portuguese and Moroccan manufactures grant Zara 60 days of trade credit when making their deals (*Tycoon Playbook*, 2013). When combining this advantage with the fact that Zara receives its cash either at the time of a retail sale, or within a week if it was a credit card purchase, you can appreciate the tremendous cash float it enjoys. Imagine we consider a firm employing QR strategy is not as powerful and affluent as Zara in the industry, like a small firm lacking sufficient bargaining power and outsourcing expediting orders from a third-party. For example, Huawei Technology Company, a Chinese networking service provider, sold the product under a relatively low bargaining power and outsourced some of the components in its early stage. It seems that he should grab more advantage of both quick response flexibility and trade credit financing if possible. Nevertheless we find this is not always the case for such a firm when the upper level manufacturer/supplier can manipulate the wholesale price in their business relationship.

Conventional wisdom tells us that with an exogenous wholesale price, the retailer will be better off under either quick response or trade credit system. Nevertheless, when the wholesale price is not given, the situation is quite different. Because of the intricacies of noncooperative Stackellberg games under uncertain demand, it is unclear which selling system would yield a higher profit for the manufacturer or the retailer in equilibrium. When comparing the equilibrium outcomes under four different systems, we formalize our understanding for several intuitive results. For example, from the manufacturer's perspective, the trade credit system benefits him most because under such a system the retailer is likely to order more in the regular period and also the manufacturer can completely control the trade credit financing by setting a proper wholesale price. In this sense, the retailer will conversely become worst under TC system. However, we also find some counterintuitive results: (a) from the manufacturer's perspective, when the retailer's relative budget is high, the manufacturer can be worst under quick credit system when compared with other three systems; and (b) from the retailer's perspective, when her relative budget is low, the traditional system will dominate all of three other selling systems; and (c) under some circumstances more ordering opportunity can be detrimental – the retailer can be worse off under TQ system than under QR system. We can attribute these unexpected results to the complexity of the decentralized structure. In addition, the spirit behind these results is in line with the findings of a two-level supply chain game under the advanced selling setting with uncertain demand and supply (Cho and Tang, 2010).

The rest of this article is organized as follows. In the next section, we review related literature. In Section 3.3, we introduce the two-level Stackellberg game under different selling systems. Section 2.4 provides the detailed analysis and solves the game and then Section 2.5 compares the manufacturer's and retailer's profits under different systems. Section 2.6 presents the numerical experiment and the managerial insights behind the analytical results. Section 2.7 extends the basic model to the case where the retailer is allowed to default. Finally, this paper is concluded in Section 2.8.

2.2. Literature Review

As discussed in the previous section, there are tremendous efforts in studying the quick response strategy in the field of operations management and we refer readers to a few research papers. Fisher and Raman (1996), Eppen and Iyer (1997), Iyer and Bergen (1997), Fisher et al. (2001) all consider the benefits of reducing supply-demand mismatches by providing the firm with an option to order inventory when the demand information is updated. Gurnani and Tang (1999) examine the case where the unit cost at the regular period is certain while this cost at the second instant is uncertain. They evaluate the tradeoff between a more accurate demand forecast and a potentially high unit cost for the second instant. Donohue (2000) investigates how retailer's quick response strategy and return policy can achieve the channel coordination. More recent works, for example, Li and Ha (2008) and Caro and Martínez-de-Albéniz (2010) address the impact of competition on quick response inventory management. Lin and Parlaktürk (2012) consider a two-retailer-one-manufacturer game when none, only one, and both of the retailers have QR ability. They show it is optimal for the manufacturer to offer QR capability to only one of the ex ante identical retailers when demand variability is sufficiently but not overly high. Cachon and Swinney (2009, 2011) do address the issue of how quick response influences the strategic consumer behavior. Serel (2012) studies a multi-product quick response system under retailer's budget constraint. But they do not consider the participation of the upstream supplier and the adoption of trade credit. For a comprehensive review on QR strategy, readers are referred to Choi and Sethi (2010).

Trade credit literature is broad and multidisiplinary, and for broad literature on economics and finance of it see Burkart and Ellingsen (2004), Giannetti et al. (2011), and reference there. In the area of operations management, we review the relevant works closest to ours that is modeled in a newsvendor setting. Zhou and Groenevelt (2007) analyze two financing schemes with credit line limits: a trade credit scheme with no early payment discount and a supplier subsidizes bank financing. In their model, the supplier's trade credit is only determined by the wholesale price and they show that the firm prefers bank credit to trade credit. Kouvelis and Zhao (2011) study different types of bankruptcy costs on the optimal wholesale price. In the following work, Kouvelis and Zhao (2012) examine the optimal trade credit scheme by choosing both interest rates and wholesale price. They show under optimal trade credit contracts, both the supplier's profit and supply chain efficiency improve and the retailer's profit might be worse when compared to the bank financing scheme. Jing et al. (2012) study the financing equilibrium between trade credit financing and bank credit financing in a channel where the retailer is capital - constrained. Jing et al. (2013) investigate the roles of bank credit and trade credit in a supply chain with a capital-constrained retailer under the case of both asymmetric information and moral hazard.

Our work is also related to the prior studies of two-level supply chain management where the manufacturer offers an only wholesale price contract to the retailer. In literature, there are hundreds of works studying this kinds of framework and here we just review two of them: Lariviere and Porteus (2001) serves a fundamental block for the development our model structure and Cho and Tang (2013) provides us the inspiration when we develop our managerial insights. Specifically, in the former one, the authors examine the determinants of the optimal wholesale price, the efficiency of the supply chain and the division of system profit; the latter work explores three selling strategies of a one-manufacturer-one-retailer supply chain under uncertain supply and demand: advance selling, regular selling and dynamic selling. They find the retailer can be worse off under regular selling than under advance selling and what's more, dynamic selling by which the retailer has a second ordering opportunity might be detrimental to her.

In this paper, we intend to study the value of quick response and/or trade credit in a two-level supply chain in which the manufacturer and the retailer engages in vertical competition and show the value to each party is not trivial under different system. We expect this finding may enrich the supply chain management literature and shed some light on further integration and development of supply chain management.

2.3. The Model

Consider a two-level supply chain that is comprised of one manufacture ("he") and one retailer ("she"). The retailer can be considered as, e.g., a fast-fashion firm who sells the products by using the material produced by the manufacturer over a short selling season. We assume the manufacturer has unlimited capacity to fulfill the retailer's demand and its unit production cost is simplified to zero. For the retailer, she is initially endowed with an on-hand budget level B, which is used for goods procurement. The market is characterized by demand uncertainty: the stochastic demand is denoted by random variable D with uniform distribution $U[0, A]$, where A is the demand scale. Here we assume the demand satisfying uniform distribution for two reasons. The first one is for tractability. Since we formulate the problem in a two-level Stackelberg setting, a general distribution assumption for this "newsvendor" game could much complicate the analysis. In order to partially generalize our results, we verify the robustness of our analytical results by using a truncated normal distribution in the numerical section. The second reason is the uniform distribution by which we can derive the desired insights has the advantage of being easily understood by the manager (Taylor, 2002).

We consider the following four selling systems:

(1) Traditional system, abbreviated T, represents a retailer can neither take advantage of a quick response strategy nor a trade credit scheme. Under this system, firstly the manufacturer charges the retailer the unit wholesale price w_T, then the retailer determines the regular ordering quantity q_T. When the selling seasons begins, the retailer sells the product at a given market price p. Note this is a typical two-level newsvendor setting under budget constraint.

(2) Trade credit system, abbreviated TC, means the retailer is granted a chance to delay her payment to the manufacturer for her regular ordering. Like the traditional one, under TC system, the retailer has only one ordering opportunity before demand realization. When the retailer actually uses the trade credit, we assume she might default on the remaining portion of the credit if her entire revenue at the end of the second season is not enough to pay back the manufacturer's loan. This is a common assumption for the trade credit literature, see (Kouvelis and Zhao, 2011) and (Bing et al., 2012). We also consider the case when the retailer is not allowed to default in the extension part. Note here we consider the optimal trade credit scheme by manufacturer's choosing only ex post wholesale price w_T (Bing et al., 2012), instead of the combination of both wholesale price and loan interest rate (Kouvelis and Zhao, 2011). This is a simplified version of trade credit contract, which actually is sufficient to manifest our insights.

(3) Quick response system, abbreviated QR, means the retailer is granted an additional ordering opportunity after demand is realized. That is, the retailer can make an expediting order from an outstanding player (e.g., a third party provider) at a fixed price e, where we assume $e < p$ representing it is profitable for this procurement. Since we do not focus on the demand updating in the second ordering period (Iyer and Bergen, 1997), we just simplify the expediting price e as a predetermined parameter. For both the regular and expediting ordering, the retailer is constrained by her initial budget B. Here we assume the third party has no long relationship with the retailer thus he requires an instant payment without any delay payment even for a known coming demand.

(4) Quick credit system, abbreviated TQ, means the retailer is able to employ both the quick-response and trade-credit flexibility. Under this system,

the retailer can not only delay the payment to the manufacturer for the regular procurement, but also own the expediting ordering chance when the demand is realized. As we mentioned in the QR system, for the expediting ordering, the retailer has to pay the expediting order by her initial endowment, rather than by any other credit or any revenue collected by the upcoming sales. In addition, we assume $B \leq Ae$, which represents the retailer's initial endowment is not enough so that she cannot order as much as she needs in the regular period, thus might rely on trade credit scheme to finance her procurement.

We model the problem as a two-level Stackelberg game, where the manufacturer sets the wholesale price as the leader and the retailer determines the order quantity as the follower. The summary of the notation is as follows:

exogenous parameters:

D = stochastic demand;

B = initial endowment owned by the retailer;

e = unit expediting ordering cost;

p = unit sales price, assume that $p > e$;

decision variables:

w = unit wholesale price;

q = regular order quantity (before the demand is realized);

2.4. Equilibrium Analysis

In this section, we use backward induction to derive subgame-perfect Nash equilibrium under different selling systems. Under each system, the manufacturer firstly sets the wholesale price w, and then the retailer makes the regular ordering decision q. Under both QR and TQ system where there is a second ordering opportunity, the retailer can make the expediting order by using her own remaining endowment.

2.4.1. Tradition System

Given the regular wholesale price w_T in the regular period, the retailer's objective is to maximize her expected profit by choosing an order quantity q_T under the budget constraint,

$$\Pi_R^T = \int_0^{q_T} (B + pD - w_T q_T)\frac{1}{A} dD + \int_{q_T}^{A} (B + pq_T - w_T q_T)\frac{1}{A} dD \quad (2.1)$$

where $q_T \leq \frac{B}{w_T}$. In the first step, we ignore the constraint of q_T and take the derivative of q_T in (2.1):

$$q_T = A(1 - \frac{w_T}{p})$$

As one would expect, without budget constraint, the retailer's regular ordering quantity q_T decreases in the wholesale price w_T. For the manufacturer, in anticipation of the retailer's ordering decision, he sets the wholesale price to maximize his expected profit:

$$\max_{w_T} \Pi_M^T = w_T q_T = w_T A(1 - \frac{w_T}{p}) \quad (2.2)$$

Then, the optimal wholesale price charged is $w_T^* = \frac{p}{2}$ and the corresponding optimal regular order quantity should be $q_T^* = \frac{A}{2}$. In addition, we derive the optimal manufacturer's profit

$$\Pi_M^{T*} = \frac{Ap}{4}$$

and the retailer's expected profit

$$\Pi_R^{T*} = B + \frac{Ap}{8}$$

The above results hold under the condition that $B \geq w_T^* q_T^* = \frac{Ap}{4}$. Denote $\frac{B}{Ap} \triangleq y$ as the measure of retailer's relative budget capability. It is observed that B is positively related to y whereas A and p are negatively related to y. Note another ratio $\frac{B}{Ae}$ can also act in a similar manner and we will use it sometimes in the later analysis. When $B < \frac{Ap}{4}$ which means the retailer is not wealthy enough to secure the optimal unconstrained regular order, she has to procure the maximal quantity by using up her on-hand account, e.g., $q_T^* = \frac{B}{w_T}$. Particularly, as long as $A(1 - \frac{w_T}{p}) \geq \frac{B}{w_T}$, the retailer will optimally order up to $q_T = \frac{B}{w_T}$ and the manufacturer can only obtain a constant profit B for such a w_T, where w_T s.t., $A(1 - \frac{w_T}{p}) \geq \frac{B}{w_T}$. Among these wholesale prices, we assume the manufacturer choose the least expensive one w_1, where w_1 is smallest in the above range, e.g., $w_1 = \frac{p}{2} - \sqrt{\frac{p^2}{4} - \frac{Bp}{A}}$. Under w_1, the retailer will benefit most and the whole supply chain achieves a Pareto optimal equilibrium.

Proposition 2.4.1. *Under traditional system (T), given retailer's relative budget* $y = \frac{B}{Ap}$, *(i) if* $\frac{B}{Ap} \leq \frac{1}{4}$, *then* $w_T^* = \frac{p}{2} - \sqrt{\frac{p^2}{4} - \frac{Bp}{A}}$, $q_T^* = \frac{A}{2} + \sqrt{\frac{A^2}{4} - \frac{AB}{p}}$, $\Pi_M^{T*} = B$ *and* $\Pi_R^{T*} = \frac{B}{2} + \frac{Ap}{4} + \frac{A}{2}\sqrt{\frac{p^2}{4} - \frac{Bp}{A}}$;
(ii) if $\frac{B}{Ap} > \frac{1}{4}$, *then* $w_T^* = \frac{p}{2}$, $q_T^* = \frac{A}{2}$, $\Pi_M^{T*} = \frac{Ap}{4}$ *and* $\Pi_R^{T*} = B + \frac{Ap}{8}$;
(iii) Π_R^{T*} *decreases in* y *when* $0 \leq y \leq \frac{1}{4}$ *and increases in* y *when* $y \geq \frac{1}{4}$.

From this proposition, it can be seen the equilibrium is divided into two parts: when y is rather small the manufacturer cannot gain more than the retailer's budget B while when y increases to a threshold, he obtains a non-constraint profit less than B. In addition, when Ap is fixed the retailer's expected profit firstly decreases in y until it reaches $\frac{1}{4}$, and then increases as y grows. This is because under the Pareto criteria when $y \leq \frac{1}{4}$, the manufacturer's optimal wholesale price increases as B grows. This effect is negative to the retailer and thus fully offsets the B's incremental effect, leading to a decreasing expected profit Π_R^{T*}. When y grows to $\frac{1}{4}$, the optimal wholesale price keeps at a constant level and the retailer's expected profit increases as y increases.

2.4.2. Trade Credit System

The main difference between the traditional system and the trade credit system lies on the budget constraint during the first period. Under TC system, the retailer will not subject to the budget constraint for the regular ordering. Since the retailer is allowed to default, she will repay the debt as much as she could at the end of horizon, e.g., $\min\{B + pD, w_{TC}q_{TC}\}$. When $B + pD < w_{TC}q_{TC}$, which means the realized revenue is not sufficient to cover the overall debt, she will claim zero capital left and leave part of her loan unpaid. Because of this, firstly we consider the case that the retailer orders a high quantity $q_{TC} > \frac{B}{w_{TC}}$ meaning she takes the advantage of trade credit financing and faces some potential default risk. Under this scenario, the retailer's expected profit can be expressed as the following,

$$\begin{aligned}\Pi_R^{TC} &= \int_0^{\frac{w_{TC}q_{TC}-B}{p}} 0\frac{1}{A}dD + \int_{\frac{w_{TC}q_{TC}-B}{p}}^{q_{TC}} (B + pD - w_{TC}q_{TC})\frac{1}{A}dD \\ &+ \int_{q_{TC}}^{A} (B + pq_{TC} - w_{TC}q_{TC})\frac{1}{A}dD \qquad (2.3)\end{aligned}$$

Using the first order condition, we have

$$q_{TC}^* = \frac{Ap(p - w_{TC}) - Bw_{TC}}{p^2 - w_{TC}^2} \tag{2.4}$$

and the manufacturer's corresponding expected profit is

$$\begin{aligned} \Pi_M^{TC} &= \int_0^{\frac{w_{TC}q_{TC}^* - B}{p}} (B + pD)\frac{1}{A}dD + \int_{\frac{w_{TC}q_{TC}^* - B}{p}}^{A} w_{TC}q_{TC}^* \frac{1}{A}dD \\ &= w_{TC}q_{TC}^* - \frac{(w_{TC}q_{TC}^* - B)^2}{2Ap} \end{aligned} \tag{2.5}$$

Notice the manufacturer's expected profit is always less than $w_{TC}q_{TC}^*$, which means he should bear some loss whenever the retailer cannot repay all the credit. Denote $w_{TC}q_{TC}^* = B + x$ and the new variable $x(\geq 0)$ represents the trade-credit line the retailer actually uses. Note x is not necessarily of one-to-one relationship with w_{TC}. Next we will consider the case $q_{TC} \leq \frac{B}{w_{TC}}$ where the firm does not turn to trade credit financing in the regular period.

Proposition 2.4.2. *Under trade credit (TC) system, (i) if* $y = \frac{B}{Ap} \leq \frac{1}{4}$, *then* $w_{TC}^* = \frac{p}{1+y+\sqrt{y^2+2y}}$, $q_{TC}^* = \frac{A}{2}$, $\Pi_M^{TC^*} = B + x_1 - \frac{x_1^2}{2Ap}$ *and* $\Pi_R^{TC^*} = B + \frac{Ap}{8} - (\frac{x_1^2}{2Ap} + \frac{Bx_1}{Ap})$, *where* $x_1 = \frac{1-y-\sqrt{y^2+2y}}{2}Ap$;
(ii) if $\frac{B}{Ap} > \frac{1}{4}$, *then* $w_{TC}^* = \frac{p}{2}$, $q_{TC}^* = \frac{A}{2}$, $\Pi_M^{TC^*} = \frac{Ap}{4}$ *and* $\Pi_R^{TC^*} = B + \frac{Ap}{8}$;
(iii) when Ap *is fixed,* $\Pi_M^{TC^*}$ *decreases in* y *while* $\Pi_R^{TC^*}$ *increases in* y.

Proof. See the Appendix. □

When the retailer is not that wealthy, e.g., $\frac{B}{Ap} \leq \frac{1}{4}$, she will optimally rely on some trade credit financing which increases the default risk if the realized demand is low. For the manufacturer who is facing this rather poor retailer, he has to shoulder all the financial risk, resulting in a relatively high wholesale price to compensate the increasing default risk. This finding is in line with the previous result that under TC system the optimal $w_{TC}^* = p$ in Jing et al. (2012). Meanwhile, it is observed that when fixed Ap the retailer's expected profit increases in y while the manufacturer's expected profit decreases in y. This can be verified from the following relationship,

$$\Pi_R^{TC^*} = B + \frac{Ap}{8} - (\frac{x_1^2}{2Ap} + \frac{Bx_1}{Ap}) = \frac{3Ap}{8} + \frac{x_1^2}{2Ap} - x_1$$
$$\Pi_M^{TC^*} = B + x_1 - \frac{x_1^2}{2Ap} = \frac{Ap}{4} + \frac{Bx_1}{Ap} + \frac{x_1^2}{2Ap},$$

where the former expected profit increases in y whereas the latter deceases in y when $y \leq \frac{1}{4}$. This is because under TC system, the manufacturer's capability to exploit the surplus is weaken when the retailer becomes richer. As the retailer becomes more wealthy, e.g., $\frac{B}{Ap} \geq \frac{1}{4}$, trade credit financing brings no effect on the retailer's ordering decision since she is not necessary to borrow from the manufacturer in the first period. Without considering trade credit financing, TC system becomes the traditional one, under which the optimal wholesale price w_{TC}^* and order quantity q_{TC}^* are independent with the retailer's budget level B. Finally from the manufacturer's perspective, under this TC system, he will earn at least the profit at the non-constrained optimal level $\frac{AP}{4}$.

2.4.3. Quick Response System

Under quick response system, the retailer is granted an additional ordering opportunity after the demand is realized by her initial budget. Given w_{QR}, her expected profit should be,

$$\begin{aligned}\Pi_R^{QR} &= \int_0^{q_{QR}} (B + pD - w_{QR}q_{QR})\frac{1}{A}dD + \int_{q_{QR}}^{q_{QR}+\frac{B-w_{QR}q_{QR}}{e}} \big(B + pD - w_{QR}q_{QR} \\ &- e(D - q_{QR})\big)\frac{1}{A}dD + \int_{q_{QR}+\frac{B-w_{QR}q_{QR}}{e}}^{A} p\big(q_{QR} + \frac{B - w_{QR}q_{QR}}{e}\big)\frac{1}{A}dD \quad (2.6)\end{aligned}$$

where $q_{QR} \leq \frac{B}{w_{QR}}$. Take the derivative of q_{QR} in (2.6),

$$q_{QR^*} = \frac{p - pw_{QR}/e - B(p-e)(e - w_{QR})/Ae^2}{e/A + (p-e)(e-w_{QR})^2/Ae^2}$$

For the manufacturer, his expected profit is

$$\max_{w_{QR}} \Pi_M^{QR} = w_{QR}q_{QR^*} = w_{QR}\frac{p - pw_{QR}/e - B(p-e)(c - w_{QR})/Ae^2}{e/A + (p-e)(e-w_{QR})^2/Ae^2}$$

Then, we derive the optimal wholesale price $w^*_{QR} = \frac{\sqrt{r}e}{\sqrt{r}+1} < e$, where $r = p/e > 1$ means the ratio of the unit sale revenue to the unit expediting cost. The higher this ratio, the more profitable when adopting the QR strategy since the expediting ordering cost becomes relatively cheap. Based on w^*_{QR}, the retailer's optimal order quantity $q^*_{QR} = \frac{A(r-(r-1)B/Ae)}{2\sqrt{r}}$ and the corresponding optimal expected profit for each player is,

$$\Pi_M^{QR^*} = \frac{Ae(r-(r-1)B/Ae)}{2(\sqrt{r}+1)}$$

$$\Pi_R^{QR^*} = \frac{(1-\sqrt{r})(\sqrt{r}+1)^2}{4\sqrt{r}}\frac{B^2}{Ae} + \frac{1}{2}(r+\sqrt{r})B + \frac{r^{3/2}}{4(\sqrt{r}+1)}Ae$$

Note for this pair of w^*_{QR} and q^*_{QR}, it requires the boundary condition $q^*_{QR} + \frac{B - w^*_{QR} q^*_{QR}}{e} \leq A$, e.g., $\frac{B}{Ae} \leq \frac{r+2\sqrt{r}}{r+2\sqrt{r}+1}$. If $\frac{B}{Ae} > \frac{r+2\sqrt{r}}{r+2\sqrt{r}+1}$, the retailer should order up to q^0_{QR}, s.t., q^0_{QR} maximizes the following,

$$\Pi^{QR}_R = \int_0^{q_{QR}} (B + pD - w_{QR}q_{QR})\frac{1}{A}dD + \int_{q_{QR}}^{A} (B + pD - w_{QR}q_{QR} - e(D - q_{QR}))\frac{1}{A}dD$$

Note we will not include this case in our following analysis since the corner solution is not interesting enough. Therefore we focus on the scenario that the retailer is not that rich: $\frac{B}{Ae} \leq \frac{r+2\sqrt{r}}{r+2\sqrt{r}+1}$. In addition, to obtain a non-capital-constrained solution, we impose the condition $B \geq w^*_{QR} q^*_{QR}$, e.g., $\frac{B}{Ap} \geq \frac{1}{r+2\sqrt{r}+1}$. Otherwise, if the on-hand cash B is not that sufficient, e.g., $\frac{B}{Ap} < \frac{1}{r+2\sqrt{r}+1}$, the retailer should order up to $\frac{B}{w_{QR}}$ in the regular period and then order nothing during the second instant. Put another way, given w_{QR}, as long as

$$\frac{p - pw_{QR}/e - B(p-e)(e - w_{QR})/Ae^2}{e/A + (p-e)(e - w_{QR})^2/Ae^2} \geq \frac{B}{w_{QR}},$$

We have $q^*_{QR} = \frac{B}{w_{QR}}$. Denote $w^*_{QR} \triangleq w_2$ as the smaller root of equation

$$\frac{p - pw_{QR}/e - B(p-e)(e - w_{QR})/Ae^2}{e/A + (p-e)(e - w_{QR})^2/Ae^2} = \frac{B}{w_{QR}} \tag{2.7}$$

For a Pareto optimal solution, the manufacturer should set the least expensive wholesale price $w^*_{QR} = w_2$. It is easy to check $w_2 > \frac{B}{A}$ so the corresponding optimal order quantity $\frac{B}{w^*_{QR}}$ does not exceed the upper bound A. Thus we have,

Proposition 2.4.3. *In this quick response (QR) system, suppose* $0 < \frac{B}{Ap} \leq \frac{1+2/\sqrt{r}}{r+2\sqrt{r}+1}$. *(i) if* $\frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$, *then* $w^*_{QR} = w_2$, $q^*_{QR} = \frac{B}{w_2}$, $\Pi^{QR*}_M = B$ *and* $\Pi^{QR*}_R = -\frac{p}{2A}(A - q^*_{QR})^2 + \frac{Ap}{2}$, *where* w_2 *is the smaller root of the equation (2.7); (ii) if* $\frac{1}{r+2\sqrt{r}+1} < \frac{B}{Ap} \leq \frac{1+2/\sqrt{r}}{r+2\sqrt{r}+1}$, *then* $w^*_{QR} = \frac{\sqrt{re}}{\sqrt{r}+1}$, $q^*_{QR} = \frac{A(r-(r-1)B/Ae)}{2\sqrt{r}}$,

$\Pi_M^{QR^*} = \frac{Ae(r-(r-1)B/Ae)}{2(\sqrt{r}+1)}$ *and* $\Pi_R^{QR^*} = \frac{(1-\sqrt{r})(\sqrt{r}+1)^2}{4\sqrt{r}}\frac{B^2}{Ae} + \frac{1}{2}(r+\sqrt{r})B + \frac{r^{3/2}}{4(\sqrt{r}+1)}Ae$; *(iii)* $\Pi_R^{QR^*}$ *decreases when* $0 < \frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$ *and then increases when* $\frac{1}{r+2\sqrt{r}+1} \leq \frac{B}{Ap} \leq \frac{1+2/\sqrt{r}}{r+2\sqrt{r}+1}$; *(iv)* $\Pi_M^{QR^*}$ *increases when* $0 < \frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$ *and then decreases when* $\frac{1}{r+2\sqrt{r}+1} \leq \frac{B}{Ap} \leq \frac{1+2/\sqrt{r}}{r+2\sqrt{r}+1}$;

When the retailer is not so rich, e.g., $y \leq \frac{1}{r+2\sqrt{r}+1}$, she should order-up-to a level by using up all her endowment during the regular period. Under this scenario, the retailer cannot make any expediting ordering as the traditional system. We leave the comparison between QR and T in the later discussion. Before y reaches the level $\frac{1}{r+2\sqrt{r}+1}$, the retailer's expected profit decreases as y grows. This is because the retailer will face a higher wholesale price while enjoy no benefit of QR capability in the second period. The higher w_{QR}^* results in a smaller q_{QR}^* and a correspondingly decreasing $\Pi_R^{QR^*}$. This finding is consistent with the decreasing $\Pi_R^{T^*}$ for $y \leq \frac{1}{4}$ under traditional system. When the retailer becomes richer, e.g., $\frac{1}{r+2\sqrt{r}+1} < \frac{B}{Ap} \leq \frac{1+2/\sqrt{r}}{r+2\sqrt{r}+1}$, she can actually adopts the expediting chance in the second period. When the relative budget increases, the advantage of demand updating in the second period eventually outweighs the inefficiency caused by the the costly expediting cost e, leading QR capability largely improves the retailer's profitability. For the manufacturer, conversely his expected profit increases at first and then decreases then. This can be interpreted as the fact that after a certain threshold a more sufficient retailer's budget is detrimental to the manufacturer since the retailer can take more advantage of the QR strategy.

2.4.4. Quick Credit System

This is a system combined with quick response flexibility and trade credit capability. Given w_{TQ} and we first assume the retailer orders a regular quantity q_{TQ}, s.t., $w_{TQ}q_{TQ} \geq B$. Then her optimal expected profit is,

$$\begin{aligned}\Pi_R^{TQ} &= \int_0^{\frac{w_{TQ}q_{TQ}-B}{p}} 0\frac{1}{A}dD + \int_{\frac{w_{TQ}q_{TQ}-B}{p}}^{q_{TQ}} (B + pD - w_{TQ}q_{TQ})\frac{1}{A}dD \\ &+ \int_{q_{TQ}}^{q_{TQ}+\frac{B}{e}} \big(B + pD - w_{TQ}q_{TQ} - e(D - q_{TQ})\big)\frac{1}{A}dD + \\ &\int_{q_{TQ}+\frac{B}{e}}^{A} \big(p(q_{TQ} + \frac{B}{e}) - w_{TQ}q_{TQ}\big)\frac{1}{A}dD \qquad (2.8)\end{aligned}$$

Using the first order condition, we have

$$q_{TQ}^* = \frac{Ap^2 - Apw_{TQ} - Bp(r-1) - Bw_{TQ}}{p^2 - w_{TQ}^2} \qquad (2.9)$$

and the manufacturer's expected profit should be,

$$\Pi_M^{TQ*} = w_{TQ}q_{TQ}^* - \frac{(w_{TQ}q_{TQ}^* - B)^2}{Ap} \qquad (2.10)$$

Proposition 2.4.4. *Under quick credit (TQ) system, suppose* $0 < \frac{B}{Ap} \leq \frac{1}{1+r}$, *(i) if* $\frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$, *then* $w_{TQ}^* = \frac{(1+y-yr)p}{1+y+\sqrt{2yr-y^2r^2+2y^2r}}$, $q_{TQ}^* = \frac{(1+y-yr)A}{2}$, $\Pi_M^{TQ^*} = B + z_1 - \frac{z_1^2}{2Ap}$ *and* $\Pi_R^{TQ^*} = \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + (\frac{1}{4} + \frac{3r}{4})B + \frac{Ap}{8}$, *where* $z_1 = \frac{1-y-\sqrt{2yr-y^2r^2+2y^2r}}{2}Ap$; *(ii) if* $\frac{1}{r+2\sqrt{r}+1} < \frac{B}{Ap} \leq \frac{1}{1+r}$, *then* $w_{TQ^*} = \frac{(1+y-yr)p}{2}$, $q_{TQ^*} = \frac{(1+y-yr)A}{2}$, $\Pi_M^{TQ^*} = \frac{A}{4p}\big(p - \frac{(p-e)B}{Ae}\big)^2$ *and* $\Pi_R^{TQ^*} = \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + (\frac{1}{4} + \frac{3r}{4})B + \frac{Ap}{8}$; *(iii)* $\Pi_M^{TQ^*}$ *decreases in* y *while* $\Pi_R^{TQ^*}$ *increases in* y.

Proof. See the Appendix. □

When the retailer is not wealthy, e.g., $y \leq \frac{1}{r+2\sqrt{r}+1}$, she will be exposed to some default risk, just as the situation she counters under TC system. In view of this, the manufacturer will set an aggressive wholesale price to compensate this financial risk, which leading to an increasing w^*_{TQ} as y grows. Even with this less expensive wholesale price, the retailer orders less in the first period due to the availability of the extra ordering opportunity. On the other hand, when $y \leq \frac{1}{r+2\sqrt{r}+1}$ it shows $w^*_{TQ} \leq w^*_{TC}$ and $q^*_{TQ} \leq q^*_{TC}$ when compared with TC system. This can be explained that both players act in a more conservative way when both QR and TC become available. In addition, as we explain for TC system, the manufacturer's capability to exploit the retailer's surplus is weaken when she becomes richer, which resulting in $\Pi_M^{TQ^*}$ decreasing in y and $\Pi_R^{TQ^*}$ increasing in y. This is in line with the property (iii) under the TC system. Furthermore we note this proposition is analogous to Proposition 2.4.3 of QR system in that under both cases the equilibrium strategy begins to change when y reaches a threshold $\frac{1}{r+2\sqrt{r}+1}$, but the meaning behind it is quite different. In Proposition 2.4.3, this level of the comparative budget ratio determines whether the retailer could actually afford the expediting ordering whereas in Proposition 2.4.4, it indicates whether the retailer might default or not under TQ system. In the next section, we further explore how each player's optimal expected profit looks like when compared with each other.

2.5. Profit Comparison under Different Selling Systems

Using the equilibrium outcomes derived in the previous section, we now compare each player's expected profits under different selling systems (T, TC, QR and TQ). The following three tables summarize each party's expected profit with different relative budget $\frac{B}{Ap}$. We assume $\frac{B}{Ap} \leq \frac{1}{1+r}$ to eliminates the corner solution as in the above analysis. Note when $r > 3$, the break-even point $\frac{1}{4}$ will occur under neither T nor TC system.

Table 2.1: Optimal Expected Profits when $\frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$

$\Pi_M^{T*} = B$	$\Pi_M^{TC*} = B + x_1 - \frac{x_1^2}{2Ap}$
$\Pi_R^{T*} = \frac{B}{2} + \frac{Ap}{4} + \frac{A}{2}\sqrt{\frac{p^2}{4} - \frac{Bp}{A}}$	$\Pi_R^{TC*} = B + \frac{Ap}{8} - \frac{x_1^2}{2Ap} - \frac{Bx_1}{Ap}$
$\Pi_M^{QR*} = B$	$\Pi_M^{TQ*} = B + z_1 - \frac{z_1^2}{2Ap}$
$\pi_R^{QR*} = -\frac{p}{2A}(A - \frac{B}{w_2})^2 + \frac{Ap}{2}$	$\Pi_R^{TQ*} = \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + (\frac{1}{4} + \frac{3r}{4})B + \frac{Ap}{8} - \frac{z_1^2}{2Ap} - \frac{Bz_1}{Ap}$

Table 2.2: Optimal Expected Profits when $\frac{1}{r+2\sqrt{r}+1} < \frac{B}{Ap} \leq \frac{1}{4}$

$\Pi_M^{T*} = B$	$\Pi_M^{TC*} = B + x_1 - \frac{x_1^2}{2Ap}$
$\Pi_R^{T*} = \frac{B}{2} + \frac{Ap}{4} + \frac{A}{2}\sqrt{\frac{p^2}{4} - \frac{Bp}{A}}$	$\Pi_R^{TC*} = B + \frac{Ap}{8} - \frac{x_1^2}{2Ap} - \frac{Bx_1}{Ap}$
$\Pi_M^{QR*} = \frac{Ae(r-(r-1)B/Ae)}{2(\sqrt{r}+1)}$	$\Pi_M^{TQ*} = \frac{A}{4p}\left(p - \frac{(p-e)B}{Ae}\right)^2$
$\Pi_R^{QR*} = \frac{(1-\sqrt{r})(\sqrt{r}+1)^2}{4\sqrt{r}}\frac{B^2}{Ae} + \frac{1}{2}(r + \sqrt{r})B + \frac{r^{3/2}}{4(\sqrt{r}+1)}Ae$	$\Pi_R^{TQ*} = \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + (\frac{1}{4} + \frac{3r}{4})B + \frac{Ap}{8}$

Table 2.3: Optimal Expected Profits when $\frac{B}{Ap} > \frac{1}{4}$

$\Pi_M^{T^*} = \frac{Ap}{4}$	$\Pi_M^{TC^*} = \frac{Ap}{4}$
$\Pi_R^{T^*} = B + \frac{Ap}{8}$	$\Pi_R^{TC^*} = B + \frac{Ap}{8}$
$\Pi_M^{QR^*} = \frac{Ae(r-(r-1)B/Ae)}{2(\sqrt{r}+1)}$	$\Pi_M^{TQ^*} = \frac{A}{4p}\left(p - \frac{(p-e)B}{Ae}\right)^2$
$\Pi_R^{QR^*} = \frac{(1-\sqrt{r})(\sqrt{r}+1)^2}{4\sqrt{r}}\frac{B^2}{Ae} + \frac{1}{2}(r+\sqrt{r})B + \frac{r^{3/2}}{4(\sqrt{r}+1)}Ae$	$\Pi_R^{TQ^*} = \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + (\frac{1}{4} + \frac{3r}{4})B + \frac{Ap}{8}$

2.5.1. Trade Credit Scheme vs. Quick Response Selling vs. Quick Credit System

In this part we compare the expected profits under TC, QR and TQ system. The following theorem summarises the results.

Theorem 2.5.1. *Suppose* $0 < \frac{B}{Ap} \leq \frac{1}{1+r}$, *(a) From the retailer's perspective,* $\Pi_R^{QR^*} \geq \Pi_R^{TQ^*} \geq \Pi_R^{TC^*}$*; (b) From the manufacturer's perspective, trade credit system dominates other two systems, and for the other two systems, (i)* $\Pi_M^{TQ^*} \geq \Pi_M^{QR^*}$ *when* $0 < \frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$*; (ii)* $\Pi_M^{QR^*} \geq \Pi_M^{TQ^*}$ *when* $\frac{1}{r+2\sqrt{r}+1} \leq \frac{B}{Ap} \leq \frac{1}{1+r}$.

Proof. See the Appendix. □

Theorem 2.5.1.(a) asserts that from the retailer's perspective, for any relative budget level $\frac{B}{Ap}$, quick response system dominates quick credit system and quick credit system dominates trade credit system. We first interpret the latter part. Compared to TC system, the retailer under TQ system can take advantage of the quick response flexibility, which obviously benefits her more. And this kind of benefit exists for any budget level due to the trade credit scheme. Next, for the former part, it seems that under TQ system the trade credit flexibility will bring her some extra advantage when compared with that under QR system. But this is not the case: for any relative budget level, the retailer will be better off under QR system than under TQ one. The underlying reason is as follows: although the retailer could enjoy the trade credit flexibility, employing the trade credit scheme under TQ system enables the manufacturer to better manipulate the wholesale price in the leader-follower game, which on the contrary eventually harms the retailer's surplus. This is an interesting finding since TQ system which combines both the quick response flexibility and trade credit financing does not necessarily benefit the retailer. The manufacturer's first-move advantage will somehow offset the potential value to the retailer under TQ system. We will see a similar result in the following discussion about the traditional system.

From the manufacturer's perspective, Theorem 2.5.1.(b) claims that trade credit system benefits the manufacturer most for any retailer's relative budget. This result is due to the following two reasons. First, under trade credit system, the retailer is likely to order more in the first period since she has no budget constraint for the regular ordering. More regular orders enable the manufacturer to extract more surplus from the retailer. Second, a selling system consisting of the quick response strategy leads the retailer to order less in the regular period because she is granted a second ordering opportunity after demand is known. We should notice this expediting ordering advantage brings no direct benefit to the manufacturer. Thus the quick response flexibility actually causes some detrimental effect to the manufacturer and he will benefit most under pure trade credit system which contains no QR flexibility. This result is consistent with the findings of Kouvelis and Zhao (2012), in which they show trade credit financing can largely improve the manufacturer's profit in a two-level supply chain structure.

It also shows when the retailer is not wealthy, e.g., $0 < \frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$, the manufacturer is better off under quick credit system than under quick response system while if the retailer becomes more wealthy, the relationship between $\Pi_M^{TQ^*}$ and $\Pi_M^{QR^*}$ is reversed. We explain this phenomenon as follows: if the retailer lacks some on-hand budget, under quick credit system the retailer will order more in the regular period than under QR system, and the manufacturer can earn more than B under TQ system; when the retailer becomes richer, this kind of advantage to the manufacturer decreases substantially under TQ system. Although the manufacturer will experience a similar profit decreasing effect under QR system, the decreasing magnitude under QR system is more slightly than that under TQ one because of a cheaper wholesale price which leading to a larger order quantity in the regular period. In this sense, whether the TQ system or the QR one benefits the manufacturer more is not straightforward, but depending on the retailer's relative budget level.

Corollary 2.5.1. *Suppose* $\frac{1}{r+2\sqrt{r}+1} < \frac{B}{Ap} \leq \frac{1}{1+r}$, *for the system's expected profit it holds* $\Pi^{QR^*} \geq \Pi^{TQ^*}$.

The proof is straightforward since $\Pi_M^{QR^*} \geq \Pi_M^{TQ^*}$ and $\Pi_R^{QR^*} \geq \Pi_R^{TQ^*}$ when $\frac{1}{r+2\sqrt{r}+1} < \frac{B}{Ap} \leq \frac{1}{1+r}$. This result sheds some managerial insights because we should pay attention that under some circumstance combining two distinctive retailer's flexibilities is beneficial to neither the manufacturer nor the retailer.

Remark: The analysis about quick response system is based on the assumption that the manufacturer seeks a Pareto equilibrium when setting the wholesale price. We can see when B is quite small the manufacturer will set a pretty low wholesale price to induce retailer's large regular order quantity. Actually, he can gain the same surplus B with a higher wholesale price. But this will hurt the retailer's welfare and results in a non-Pareto equilibrium, so we should eliminate this scenario in our analysis.

2.5.2. The Influence of Traditional System

Now we compare the expected profit under traditional system with those under other three systems.

Theorem 2.5.2. *Suppose* $0 < \frac{B}{Ap} \leq \frac{1}{1+r}$ *and let* $l = \min\{\frac{1}{1+r}, \frac{1}{4}\}$. *(a) From the retailer's perspective, it holds (i)* $\Pi_R^{T*} \geq \max\{\Pi_R^{TC*}, \Pi_R^{QR*}, \Pi_R^{TQ*}\}$ *when* $0 < \frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$*; (ii)* $\Pi_R^{T*} \geq \Pi_R^{TC*}$ *when* $\frac{1}{r+2\sqrt{r}+1} \leq \frac{B}{Ap} \leq l$*; (iii)* $\Pi_R^{QR*} \geq \Pi_R^{TQ*} \geq \Pi_R^{TC*} = \Pi_R^{T*}$ *when* $\frac{B}{Ap} \geq \frac{1}{4}$. *(b) From the manufacturer's perspective, (i)* $\Pi_M^{TC*} \geq \Pi_M^{TQ*} \geq \Pi_M^{QR*} = \Pi_M^{T*}$ *when* $0 < \frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$*; (ii)* $\Pi_M^{TC*} \geq \Pi_M^{T*} \geq \Pi_M^{QR*} \geq \Pi_M^{TQ*}$ *when* $\frac{1}{r+2\sqrt{r}+1} \leq \frac{B}{Ap} \leq l$*; (iii)* $\Pi_M^{TC*} = \Pi_M^{T*} \geq \Pi_M^{QR*} \geq \Pi_M^{TQ*}$ *when* $\frac{B}{Ap} \geq \frac{1}{4}$.

Proof. See the Appendix. □

Theorem 2.5.2. is supplementary to the previous theorem when taking into account the traditional system. Part (b) about the manufacturer's expected profit is under expectation: when $\frac{B}{Ap}$ is small, under traditional system the manufacturer can only obtain the retailer's full endowment B, which equals to that under QR system but is smaller than that under TC and TQ system; when $\frac{B}{Ap}$ increases to a certain level $\frac{1}{r+2\sqrt{r}+1}$, the manufacturer can still earn B under the traditional system however he is not able to extract the whole retailer's endowment B under either QR or TQ system. This is because the quick response flexibility will weaken the manufacturer's first-move advantage, which leading to a larger retailer's surplus under T system, e.g., $\max\{\Pi_M^{QR*}, \Pi_M^{TC*}\} \leq \Pi_M^{T*}$; when $\frac{B}{Ap}$ further increases, the traditional system is identical to the TC system, under which the manufacturer can benefit most as we asserted before. As for the retailer's part, the result also relies on the Pareto-optimal prerequisite. Under this assumption, the analysis of the traditional system is not so intuitive as we expect: when the relative budget is low. e.g., $0 < \frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$, the traditional system will dominate all of the other three selling systems. Recall we have shown $\Pi_R^{QR^*} \geq \Pi_R^{TQ^*} \geq \Pi_R^{TC^*}$ and the question left is why the retailer will prefer T system to QR one. Given manufacturer's knowledge that he will optimally earn the full endowment B, the retailer's potential flexibility under QR system will eventually result in an more aggressive wholesale price. Therefore similar to Theorem 2.5.1.(a) where the retailer will be worse off under TQ system than under QR system, more flexibility can be detrimental to the retailer when compared to the traditional system where there is no flexibility! When $\frac{B}{Ap}$ gets larger but less than l, the situation is much complicated, the T system may outweigh the QR system or might be inferior to the TQ system. Here are two examples:

Example 2.5.1. *Consider an instance with* $y = \frac{B}{Ap} = 0.18$ *and* $r = \frac{p}{c} = 2$*, then*

$$
\begin{aligned}
& \Pi_R^{T^*} \geq \Pi_R^{QR^*} \\
\iff & \frac{1}{2}(r+\sqrt{r}-1)y < \frac{1}{4(\sqrt{r}+1)} + \frac{1}{2}\sqrt{\frac{1}{4}-y} + \frac{\sqrt{r}(\sqrt{r}-1)(\sqrt{r}+1)^2}{4}y^2 \\
\iff & 0.046 \geq 0
\end{aligned}
$$

which follows the relationship $\Pi_R^{T^*} \geq \Pi_R^{QR^*} \geq \Pi_R^{TQ^*}$.

Example 2.5.2. *Consider another instance with* $y = \frac{1}{4}$ *and* $r = 2$*, at this time,*

$$
\begin{aligned}
\Pi_R^{TQ^*} \geq \Pi_R^{T^*} \iff & \frac{3r-1}{4}y - \frac{3r^2-2r-1}{8}y^2 - \frac{1}{2}\sqrt{\frac{1}{4}-y} - \frac{1}{8} \geq 0 \\
\iff & \frac{17}{128} \geq 0
\end{aligned}
$$

which implies $\Pi_R^{QR^*} \geq \Pi_R^{TQ^*} \geq \Pi_R^{T^*}$.

We can see when the retailer is not rich ($y = 0.18$), the traditional system might outweigh the quick response one since at this time the quick response flexibility cannot bring much benefit to the retailer due to the insufficient budget and potential Pareto requirement under the traditional system. However if the retailer becomes richer to a certain level ($y = 0.25$), even under the trade credit system could the retailer gain more than that under the traditional system since the flexibility could bring some actual benefit to the retailer. Furthermore when y continues to increases, the trade credit scheme cannot affect the retailer's decision and TC system is identical to T system.

2.6. Numerical Experiment

We now present numerical experiments to gain further insights with regard to the manufacturer's, the retailer's and the overall system's expected profit under different selling systems. In our illustration, we fix the demand scale $A = 5$, expediting ordering cost $e = 1$ and vary the initial endowment B. We examine two kinds of demand distribution for several instances, one is uniform distribution $D \sim U[0, A]$ and the other is truncated normal distribution $D \sim N(A/2, 2^2)|[0, A]$. Under each demand distribution, we assume $p = 2$ and $p = 4$ respectively to represent how "rich" the retailer is, e.g., $p = 2$ ($p = 4$) representing the retailer is relatively rich (poor) and we vary the initial endowment B for a certain interval with an increment of 0.1(0.3) for uniform (truncated normal) demand setting. Also we use "T", "C", "R" and "Q" to represent each system respectively in the following illustration. As seen in Figure 2.1 and 2.2, under both demand distributions, the numerical experiment verifies our analytical results: (a) as B grows, the manufacturer's expected profit decreases under

TC and TQ system, increases under T system and first increases then decreases under QR system; and (b) given a certain threshold $t = \frac{Ap}{r+2\sqrt{r+1}}$, when $B \leq t$, $\Pi_M^{TC*} \geq \Pi_M^{TQ*} \geq \Pi_M^{QR*} = \Pi_M^{T*}$ and $\Pi_M^{TC*} \geq \Pi_M^{T*} \geq \Pi_M^{TQ*} \geq \Pi_M^{QR*}$ when $B > t$. We also find within our budget interval ($B \sim [0, 4]$), Π_M^{T*} will converge to Π_M^{TC*} when $p = 2$ while for $p = 4$, Π_M^{T*} is much smaller than Π_M^{TC*} for the whole budget range. This is because when $p = 4$, the initial endowment to afford the non-constrained optimal regular ordering is larger than that when $p = 2$, which leads to $q_M^{TC*} > q_M^{T*}$ and $\Pi_M^{TC*} > \Pi_M^{T*}$ correspondingly.

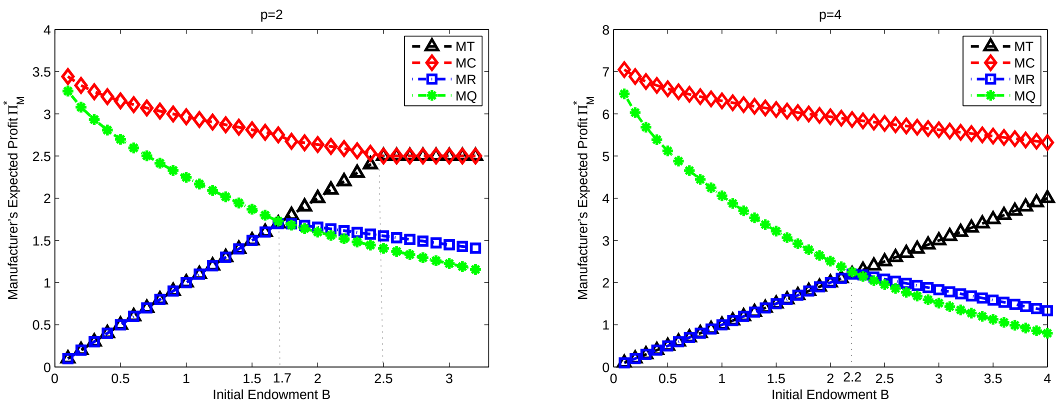

Figure 2.1: Manufacturer's Expected Profit with respect to Budget Level B

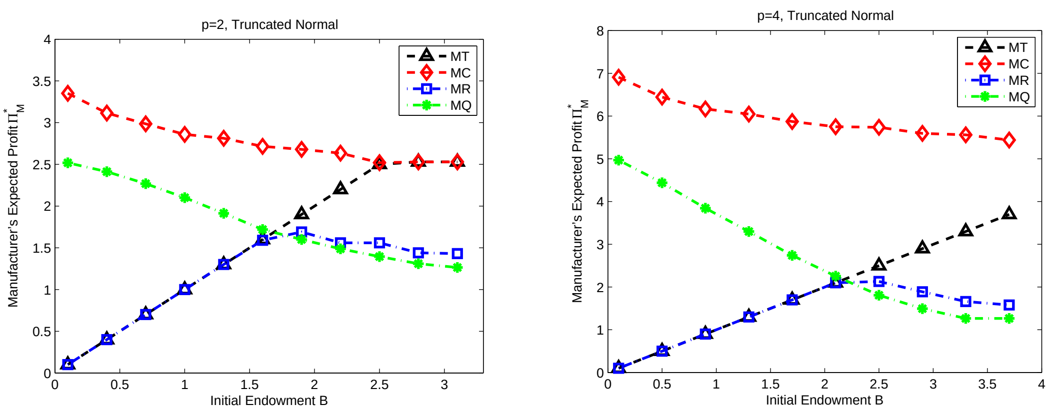

Figure 2.2: Manufacturer's Expected Profit with respect to Budget Level B

In Figure 2.3 and 2.4, the following observations confirm most of our analytical conclusion: (c) as B grows, the retailer's expected profit increases under both TC and TQ systems, and it first decreases until t and then increases under T and QR systems; and (d) $\Pi_R^{T^*} \geq \Pi_R^{QR^*} \geq \Pi_R^{TQ^*} \geq \Pi_R^{TC^*}$ when $B \leq c$ and $\Pi_R^{QR^*}(\Pi_R^{TQ^*}) \geq \Pi_R^{T^*} \geq \Pi_R^{TC^*}$ when $B > t$. The first part of (d) is the characterization of Example 2.5.1. in which T system outweighs other three systems under the case of low budget level. This echoes the fact that the retailer will be better off by using traditional system when she is not rich and prefer to quick response or trade credit when she gets wealthy. We observe the relationship $\Pi_R^{QR^*} \geq \Pi_R^{TQ^*}$ does not hold in the truncated normal case where it occurs $\Pi_R^{QR^*} < \Pi_R^{TQ^*}$ for some large B. This may be due to the differentiation between the two demand distributions. In addition, we find even under the uniform case where $\Pi_R^{QR^*} \geq \Pi_R^{TQ^*}$ for large B, e.g., $B = 2.5$, the magnitude of the profit difference is rather small, representing the fact that the retailer can obtain almost the same surplus under QR system as that under TQ one. In this sense, when B grows to a large level, QR system and TQ system benefit the retailer nearly to the same extent.

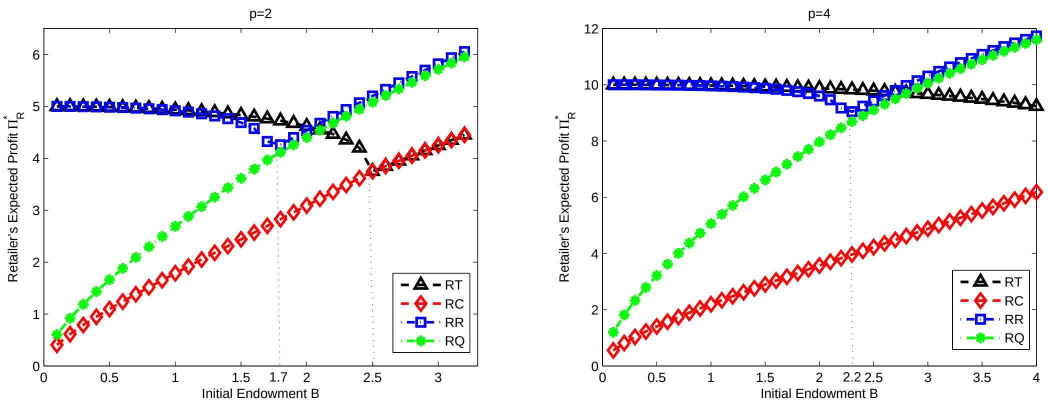

Figure 2.3: Retailer's Expected Profit with respect to Budget Level B

In Figure 2.5 and 2.6, we examine the system's expected profit and draw the

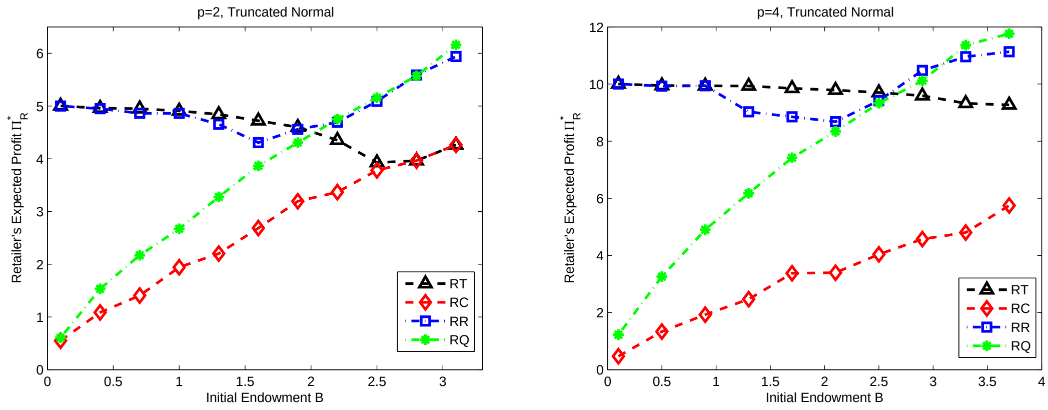

Figure 2.4: Retailer's Expected Profit with respect to Budget Level B

following observations: (e) when $p = 2$ and as B increases, Π^{T^*} and Π^{QR^*} first decreases and then increases while Π^{TC^*} and Π^{TQ^*} increase all the time; when $p = 4$, Π^{QR^*} first decreases and then increases while the other three system's expected profit increase all the time; and (f) under uniform distribution, when $p = 2$, $\Pi^{T^*} \geq \Pi^{QR^*} \geq \Pi^{TQ^*} \geq \Pi^{TC^*}$ for a small B, e.g., $B = 2.4$ and $\Pi^{QR^*} \geq \Pi^{TQ^*} \geq \Pi^{TC^*} = \Pi^{T^*}$ when B exceeds this certain level 2.4; when $p = 4$, it follows $\Pi^{T^*} \geq \Pi^{QR^*} \geq \Pi^{TQ^*} \geq \Pi^{TC^*}$ for every B in the given interval; and (g) under truncated normal distribution, most conclusions in (f) follow except that (1) $\Pi^{TC^*} > \Pi_R^{TQ^*}$ for small B and (2) when $p = 4$, the comparison between Π^{T^*}, Π^{QR^*} and Π^{TQ^*} is ambiguous for large B and actually they are nearly the same when $B \in [3.3, 4]$.

From these observations, we can conclude that QR or TQ system only benefits the whole supply chain when the initial budget suffices to afford the regular ordering, otherwise the system performs best under the traditional system. TC system will not benefit the system much when compared with the other three. The reason is two-fold: firstly the retailer's incentive to order will greatly be distorted by the manufacturer's first-move advantage and the second one is the system could achieve a Pareto optimal expected profit under both T and QR system even given retailer's insufficient endowment, which outweighs the benefit under TC system .

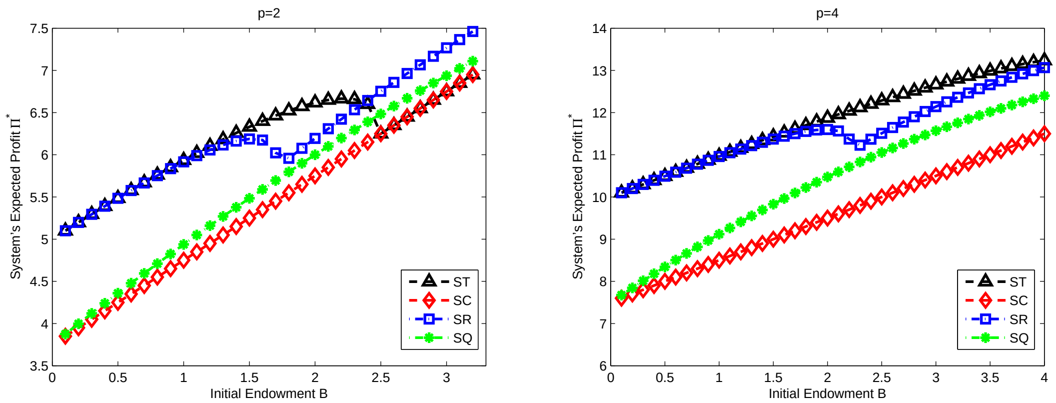

Figure 2.5: System's Expected Profit with respect to Budget Level B

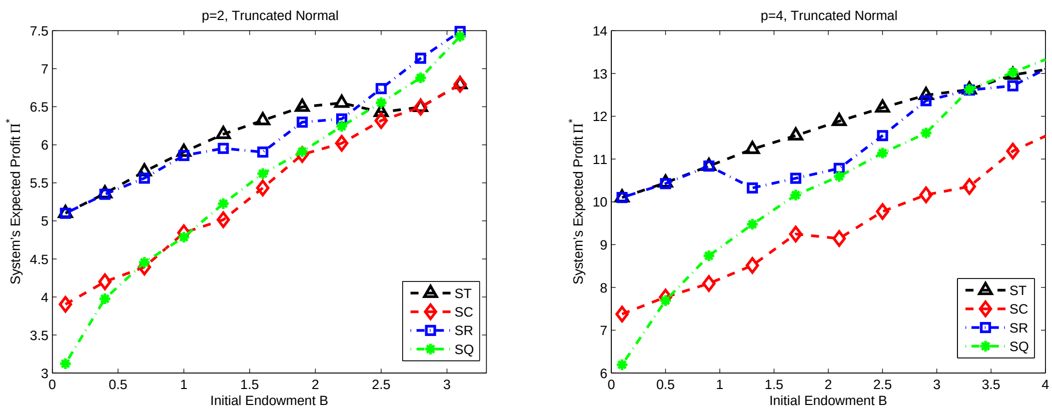

Figure 2.6: Manufacturer's Expected Profit with respect to Budget Level B

2.7. Retailer's Non-default Case

In this section, we consider a case where the retailer is not allowed to default when using trade credit scheme. This occurs when the retailer has some kinds of assets or collateral to liquidate but these assets cannot be converted to cash for the regular ordering. Specifically, under TC system, even when the terminal revenue is not sufficient to cover the loan, she should pay the difference by liquidating the assets and denote a negative amount on her accounting lodger. Recall we assume a zero expected profit when default occurs in the former case. In this sense, the retailer's expected profit is,

$$\pi_R^{TC} = \int_0^{q_{TC}} (B + pD - w_{TC}q_{TC})\frac{1}{A}dD + \int_{q_{TC}}^{A} (B + pq_{TC} - w_{TC}q_{TC})\frac{1}{A}dD$$

and the manufacturer should always collect $\pi_M^{TC} = w_{TC}q_{TC^*}$ in this case. Similarly, under TQ system, we have,

$$\begin{aligned}\pi_R^{TQ} &= \int_0^{q_{TQ}} (B + pD - w_{TQ}q_{TQ})\frac{1}{A}dD \\ &+ \int_{q_{TQ}}^{q_{TQ}+\frac{B}{e}} \big(B + pD - w_{TQ}q_{TQ} - e(D - q_{TQ})\big)\frac{1}{A}dD + \\ &\int_{q_{TQ}+\frac{B}{e}}^{A} \big(p(q_{TQ} + \frac{B}{e}) - w_{TQ}q_{TQ}\big)\frac{1}{A}dD\end{aligned}$$

and

$$\pi_M^{TQ} = w_{TQ}q_{TQ^*}$$

By the same reasoning as the default case, we have two parallel propositions,

Proposition 2.7.1. *Under trade credit system when the retailer is not allowed to default, it follows* $w_{TC^*} = \frac{p}{2}$, $q_{TC^*} = \frac{A}{2}$, $\pi_M^{TC^*} = \frac{Ap}{4}$ *and* $\pi_R^{TC^*} = B + \frac{Ap}{8}$.

Proposition 2.7.2. *Under quick credit system when the retailer is not allowed to default,* $w_{TQ^*} = \frac{(1+y-yr)p}{2}$, $q_{TQ^*} = \frac{(1+y-yr)A}{2}$, $\pi_M^{TQ^*} = \frac{A}{4p}\big(p - \frac{(p-e)B}{Ae}\big)^2$ *and* $\pi_R^{TQ^*} = \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + (\frac{1}{4} + \frac{3r}{4})B + \frac{Ap}{8}$.

Furthermore, compared with the performance between TC and TQ system where the retailer is allowed to default, we have,

(1) $\pi_R^{TC^*} \geq \Pi_R^{TC^*}$ and $\pi_M^{TC^*} \leq \Pi_M^{TC^*}$, particularly $\pi_R^{TC^*} > \Pi_R^{TC^*}$ and $\pi_M^{TC^*} < \Pi_M^{TC^*}$ when $\frac{B}{Ap} \leq \frac{1}{4}$;

(2) $\pi_R^{TQ^*} \geq \Pi_R^{TQ^*}$ and $\pi_M^{TQ^*} \leq \Pi_M^{TQ^*}$, particularly $\pi_R^{TQ^*} > \Pi_R^{TQ^*}$ and $\pi_M^{TQ^*} < \Pi_M^{TQ^*}$ when $\frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$.

When the retailer is not rich enough, e.g., $\frac{B}{Ap} \leq \frac{1}{4}(\frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1})$ under TC (TQ) system, she will be strictly better off under the non-default case while the manufacture will on the contrary be worse off under this case. This is not straightforward since the manufacturer might take more risk when facing retailer's default possibility. The reason is when the retailer is allowed to default which means she is likely to order more in the first period, the manufacturer therefore will charge a higher wholesale price to compensate the financial risk while still inducing the same retailer's regular order quantity $q_{TC^*}(q_{TQ^*})$ under TC (TQ) system. In this sense, the retailer's default scheme can actually help the manufacturer to extract more benefit from the retailer's surplus. When the retailer is getting richer, these two scheme will generate the same surplus for both parties. Numerically, we find the comparison between each player's expected profit under different system is analogous to the pervious non-default case.

2.8. Concluding Remarks

In this paper, we have examined four different selling strategies (traditional, quick-response, trade-credit, quick-credit) where a manufacturer who sells a seasonal product through a retailer under budget constraint and demand uncertainty. We model these strategies as different Stackelberg games in which the random demand follows a uniform distribution and we characterize the equilibrium for each system.

From the retailer's perspective, we show that the quick response system dominates quick credit system and quick credit system dominates trade credit system. In addition, it follows that the retailer will be better off under traditional system than under other three systems when she is not wealthy. These results contradict our conventional wisdom since the retailer should be better off under a system with more flexibility. We could attribute this findings to the manufacturer's control of wholesale price. The first-move advantage of the control enables the manufacturer to extract more benefit from the retailer under either quick response or trade credit flexibility. In particular, the trade credit financing strengthens this first-move advantage which weakens the retailer's benefit, whereas the quick response strategy does the opposite.

From the manufacturer's perspective, we find the trade credit system dominates other three systems and the traditional system could outweigh either the quick response or quick credit system. This again echoes the fact that trade credit financing can strengthen the manufacturer's first-move advantage whereas the retailer's quick response flexibility might weaken this kind of advantage, thus resulting an inferior outcome for the manufacturer under either quick response or quick credit system.

In addition, we explore an extended case where the retailer is not allowed to default and find most profit comparison results for the default case still hold. As to the whole supply chain system, by extensive numerical experiments we find either quick response or quick credit system can benefit the whole supply chain most when the initial budget is not too low while for the relatively low retailer's budget level the system may performs best under the traditional system.

Our models have some limitations from the assumptions such as uniformly distributed demand and the complete information. Relaxing these assumptions could possibly lead to new insights, which, however, will also result in significant analytical challenges. We leave it for our future research.

2.9. Appendix

Proof of Proposition 2.4.2

Proof. Plugging $q^*_{TC} = \frac{B+x}{w_{TC}}$ into (2.4), it holds,

$$(Ap - x)w_{TC}^2 - Ap^2 w_{TC} + (B + x)p^2 = 0 \tag{2.11}$$

and into (2.5),

$$\Pi_M^{TC} = B + x - \frac{x^2}{2Ap} \tag{2.12}$$

In order to ensure the existence of w_{TC} in (2.12), x should satisfy,

$$A^2p^4 - 4(Ap - x)(B + x)p^2 = m^2 \geq 0 \tag{2.13}$$

where $m \geq 0$. There are two cases for the further analysis:

Case 1: $0 \leq \frac{B}{Ap} \leq \frac{1}{4}$

Under this case, (2.13) requires $0 < x \leq x_1 \leq \frac{Ap-B}{2}$ or $Ap \geq x \geq x_2 \geq \frac{Ap-B}{2}$ where x_1 and x_2 are the two positive roots for $m = 0$, e.g., $x_1 = \frac{1-y-\sqrt{y^2+2y}}{2}Ap$ and $x_2 = \frac{1-y+\sqrt{y^2+2y}}{2}Ap$. If $x \geq x_2$, it requires the retailer's remaining revenue should exceed the initial endowment B, otherwise the retailer will not join this game. When $x \geq x_2$, the corresponding $w_{TC} = \frac{Ap^2+m}{2(Ap-x)}$ and $q_{TC} = \frac{B+x}{w_{TC}}$, and plugging them into (2.11),

$$\begin{aligned} \Pi_R^{TC}(x) &= \frac{x^2}{2Ap} - x + pq_{TC} - \frac{pq_{TC}^2}{2A} \\ &= \frac{x^2}{2Ap} - x + \frac{(Ap^2-m)(3Ap^2+m)}{8Ap^3} \end{aligned} \tag{2.14}$$

Notice (2.14) decreases in x when $\frac{Ap-B}{2} \leq x \leq Ap$. Hence if we require $\Pi_R^{TC}(x) \geq B$, it follows

$$\Pi_R^{TC}(x_2) = B + \frac{Ap}{8} - \frac{x_2^2}{2Ap} - \frac{Bx_2}{Ap} \geq B \tag{2.15}$$

which is equivalent to the following inequity,

$$\begin{aligned} &\frac{x_2^2}{2Ap} + \frac{Bx_2}{Ap} \leq \frac{Ap}{8} \\ &\iff (1+y)\sqrt{y^2+2y} \leq y^2 - 2y \end{aligned} \tag{2.16}$$

When $y \leq \frac{1}{4}$, the above inequity never holds since $(1+y)\sqrt{y^2+2y} \geq 0$ and $y^2 - 2y \leq 0$, which indicates the assumption $x \geq x_2$ does not hold. Therefore the feasible x should be less than x_1. Under such an assumption, we can show $\Pi_R^{TC}(x_1) \geq B$:

$$\begin{aligned} &\frac{x_1^2}{2Ap} + \frac{Bx_1}{Ap} \leq \frac{Ap}{8} \\ &\iff -(1+y)\sqrt{y^2+2y} \leq y^2 - 2y \quad (y^2 - 2y \leq 0) \\ &\iff (1+y)^2(y^2+2y) \geq (y^2-2y)^2 \end{aligned} \tag{2.17}$$

Note in (2.12), Π_M^{TC} is increasing in x when $x \leq Ap$. Therefore the manufacturer should choose the maximal feasible x_1 to achieve the optimal surplus, which also ensures the participation of the retailer.

Case 2: $\frac{B}{Ap} > \frac{1}{4}$

Under this case, there exists only one positive root $x_2 = \frac{1-y+\sqrt{y^2+2y}}{2}$ satisfying (2.13). For $x \geq x_2$, by the same argument as in Case 1, it can be proved that

$$\begin{aligned} & \frac{x_2^2}{2Ap} + \frac{Bx_2}{Ap} > \frac{Ap}{8} \\ & \Longleftrightarrow (1+y)\sqrt{y^2+2y} \geq y^2 - 2y \end{aligned}$$

which leads to $\Pi_R^{TC}(x) \leq \Pi_R^{TC}(x_2) = B + \frac{Ap}{8} - \frac{x_2^2}{2Ap} - \frac{Bx_2}{Ap} \leq B$. Thus we claim that the manufacturer cannot set a positive x because with this $x(w_{TC})$, the retailer can only gain an expected profit less than her initial budget B, which is unacceptable for her. In view of this, the manufacturer should choose a mild w_{TC}, s.t., $w_{TC}q_{TC}^* \leq B$, leading the retailer's regular ordering without any trade credit financing:

$$\Pi_R^{TC} \quad = \quad \int_0^{q_{TC}} (B + pD - w_{TC}q_{TC})\frac{1}{A}dD + \int_{q_{TC}}^{A} (B + pq_{TC} - w_{TC}q_{TC})\frac{1}{A}dD$$

Under this traditional scenario, the problem can be easily solved. □

Proof of Proposition 2.4.4

Proof. Denote $w_{TQ}q_{TQ}^* = B + z$, where $z \geq 0$ and note this new variable z is not necessarily of one-to-one relationship with w_{TQ}. plugging $q_{TQ}^* = \frac{B+z}{w_{TQ}}$ into (2.9),

$$(Ap - z)w_{TQ}^2 + (Bpr - AP^2 - Bp)w_{TQ} + (B + z)p^2 = 0 \tag{2.18}$$

and into (2.10)

$$\Pi_M^{TQ^*} = B + z - \frac{z^2}{2Ap} \tag{2.19}$$

To ensure the existence of w_{TQ}, z should satisfy

$$g(z) = z^2 + (B - AP)z + \frac{1}{4}(Ap + B - Br)^2 - ABp \geq 0 \tag{2.20}$$

Case 1: $\frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$

Under this case, (2.20) requires $0 < z \leq z_1 \leq \frac{Ap-B}{2}$ or $z > z_2 > \frac{Ap-B}{2}$ where z_1 and z_2 are the two positive roots for equation $g(z) = 0$, e.g., $z_1 = \frac{1-y-\sqrt{2yr-y^2r^2+2y^2r}}{2}Ap$ and $z_2 = \frac{1-y+\sqrt{2yr-y^2r^2+2y^2r}}{2}Ap$. If $z \geq z_2$, to ensure the retailer to participate, it requires,

$$\Pi_R^{TQ}(z_2) = \frac{-3r^2 + 2r + 1}{8}\frac{B^2}{Ap} + (\frac{1}{4} + \frac{3r}{4})B + \frac{Ap}{8} - \frac{z_2^2}{2Ap} - \frac{Bz_2}{Ap} \geq B$$

which is equivalent to the following inequity,

$$(y + 1)\sqrt{2yr - y^2r^2 + 2y^2r} \leq -y^2r^2 + 2y^2 - 2y + 2yr \tag{2.21}$$

But

$$(y+1)\sqrt{2yr - y^2r^2 + 2y^2r} \geq -y^2r^2 + 2y^2r + 2yr \geq -y^2r^2 + 2y^2 - 2y + 2yr$$

which is contradictory to (2.21). Therefore the feasible z should be less than z_1 and we can show $\Pi_R^{TQ}(z_1) \geq B$:

$$\begin{aligned} \Pi_R^{TQ}(z_1) &= \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + (\frac{1}{4}+\frac{3r}{4})B + \frac{Ap}{8} - \frac{z_1^2}{2Ap} - \frac{Bz_1}{Ap} \geq B \\ \Longleftrightarrow & \quad -(y+1)\sqrt{2yr - y^2r^2 + 2y^2r} \leq -y^2r^2 + 2y^2 - 2y + 2yr \end{aligned}$$

Proof. If $1 \leq r \leq \sqrt{2}$, $-y^2r^2+2y^2-2y+2yr \geq 0 \geq -(y+1)\sqrt{2yr - y^2r^2 + 2y^2r}$; Otherwise if $r > \sqrt{2}$, it holds $-yr^2 + 2y - 2 + 2r \geq 2r - 2 - \frac{r^2}{r+2\sqrt{r}+1} = \frac{r^2+4r\sqrt{r}-4\sqrt{r}-2}{r+2\sqrt{r}+1} \geq 0$, which also indicates $-y^2r^2 + 2y^2 - 2y + 2yr \geq 0 \geq -(y+1)\sqrt{2yr - y^2r^2 + 2y^2r}$. □

So the manufacturer should choose this optimal $z_1(w_{TQ})$ to ensure the participation of retailer.

Case 2: $\frac{B}{Ap} > \frac{1}{r+2\sqrt{r}+1}$

Under this case, there exists only one positive z_2 satisfying (2.20). For this $z \geq z_2$, it can be proved that $\Pi_R^{TQ}(z_2) < B$:

$$(y+1)\sqrt{2yr - y^2r^2 + 2y^2rx} \geq -y^2r^2 + 2y^2 - 2y + 2yr$$

Hence at this time the manufacturer can never earn a profit more than B. So we consider the case $w_{TQ}q_{TQ}^* \leq B$, under which the expected retailer's profit becomes the following case,

$$\begin{aligned}\Pi_R^{TQ} &= \int_0^{q_{TQ}} (B+pD-w_{TQ}q_{TQ})\frac{1}{A}dD + \int_{q_{TQ}}^{q_{TQ}+\frac{B}{e}} \big(B+pD-w_{TQ}q_{TQ} \\ &\quad -e(D-q_{TQ})\big)\frac{1}{A}dD + \int_{q_{TQ}+\frac{B}{e}}^{A} \big(p(q_{TQ}+\frac{B}{e})-w_{TQ}q_{TQ}\big)\frac{1}{A}dD\end{aligned}$$

we have,

$$\begin{aligned} w_{TQ^*} &= \frac{(1+y-yr)p}{2}, \qquad q_{TQ^*} = \frac{(1+y-yr)A}{2}, \\ \Pi_M^{TQ^*} &= \frac{A}{4p}\big(p-\frac{(p-e)B}{Ae}\big)^2, \qquad \Pi_R^{TQ^*} = \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + (\frac{1}{4}+\frac{3r}{4})B + \frac{Ap}{8}\end{aligned}$$

in order to guarantee to be an inner optimal solution, it should require $q_{TQ^*} + \frac{B}{e} \leq A$, which equivalent to the condition $\frac{B}{Ap} \leq \frac{1}{1+r}$ or $\frac{B}{Ae} \leq \frac{r}{1+r}$. □

Proof of Theorem 2.5.1 and 2.5.2

Proof. **Case 1:** when $\frac{B}{Ap} \leq \frac{1}{r+2\sqrt{r}+1}$

for the manufacturer it follows

$$\Pi_M^{TC^*} = B + x_1 - \frac{x_1^2}{2Ap} \geq \Pi_M^{TQ^*} = B + z_1 - \frac{z_1^2}{2Ap} \geq \Pi_M^{QR^*} = \Pi_M^{T^*} = B$$

It suffices to show

$$\begin{aligned}
& \Pi_M^{TC^*} \geq \Pi_M^{TQ^*} \\
\iff\quad & B + x_1 - \frac{x_1^2}{Ap} \geq B + z_1 - \frac{z_1^2}{Ap} \\
\iff\quad & x_1 = \frac{1 - y - \sqrt{y^2 + 2y}}{2} Ap \geq z_1 = \frac{1 - y - \sqrt{2yr - y^2r^2 + 2y^2r}}{2} Ap \\
\iff\quad & y \leq \frac{2}{r-1} \\
\iff\quad & \frac{1}{r + 2\sqrt{r} + 1} \leq \frac{2}{r-1}
\end{aligned}$$

for the retailer, first it holds

$$\begin{aligned}
& \Pi_R^{TQ^*} \geq \Pi_R^{TC^*} \\
\iff\quad & \frac{-3r^2 + 2r + 1}{8} \frac{B^2}{Ap} + \frac{3r+1}{4} B + \frac{Ap}{8} - \frac{z_1^2}{2Ap} - \frac{Bz_1}{Ap} \geq B + \frac{Ap}{8} - \frac{x_1^2}{2Ap} - \frac{Bx_1}{Ap} \\
\iff\quad & \frac{-3r^2 + 2r + 1}{8} \frac{B^2}{Ap} + \frac{3r+1}{4} B + \frac{Ap}{8} \geq B + \frac{Ap}{8} \quad (x_1 \geq z_1) \\
\iff\quad & \frac{r-1}{4} B (3 - \frac{3r+1}{2r} \frac{B}{Ae}) \geq 0
\end{aligned}$$

and,

$$\begin{aligned}
& \Pi_R^{QR^*} \geq \Pi_R^{TQ^*} \\
\iff\quad & -\frac{p}{2A}(A - \frac{B}{w_2})^2 + \frac{Ap}{2} \\
= \quad & \frac{Ap}{8}\left(1 + yr - y + \sqrt{(1 + yr - y)^2 - 4yr}\right)\left(3 - yr + y - \sqrt{(1 + yr - y)^2 - 4yr}\right) \\
\geq \quad & \frac{-3r^2 + 2r + 1}{8} \frac{B^2}{Ap} + \frac{3r+1}{4} B + \frac{Ap}{8} \\
\iff\quad & 2(1 + y - yr)\sqrt{(1 + yr - y)^2 - 4yr}) + (r^2 + 2r - 3)y^2 - (2r + 2)y + 1 \geq 0 \\
\iff\quad & (r^2 + 2r - 3)y^2 - (2r + 2)y + 1 \geq 0 \qquad (2.22)
\end{aligned}$$

Note $0 \leq y \leq \frac{1}{r+2\sqrt{r}+1}$ and $\frac{1}{r+2\sqrt{r}+1} \leq \frac{r+1}{r^2+2r-3}$, (2.22) follows when $y \leq \frac{1}{r+2\sqrt{r}+1}$. finally,

$$\begin{aligned} & \Pi_R^{T^*} \geq \Pi_R^{QR^*} \\ \Longleftrightarrow \quad & -\frac{p}{2A}(A - \frac{B}{w_1})^2 + \frac{Ap}{2} \geq -\frac{p}{2A}(A - \frac{B}{w_2})^2 + \frac{Ap}{2} \\ \Longleftrightarrow \quad & q_T^* = \frac{A}{2}(1 + \sqrt{1-4y}) \geq q_{QR}^* = \frac{A}{2}\big(1 + yr - y + \sqrt{(1+yr-y)^2 - 4yr}\big) \end{aligned}$$

Case 2: $\frac{1}{r+2\sqrt{r}+1} \leq \frac{B}{Ap} \leq \frac{1}{4}$

for the manufacturer it follows,

$$\Pi_M^{TC^*} \geq \frac{Ap}{4} \geq \Pi_M^{T^*} = B \geq \Pi_M^{QR^*} = \frac{Ae\big(r - (r-1)\frac{B}{Ae}\big)}{2(\sqrt{r}+1)} \geq \Pi_M^{TQ^*} = \frac{A}{4p}\Big(p - \frac{(p-e)B}{Ae}\Big)^2$$

for the retailer, the relationship $\Pi_R^{T^*} \geq \Pi_R^{TC^*}$ and $\Pi_R^{TQ^*} \geq \Pi_R^{TC^*}$ can be proved by the same argument as in Case 1. Here we just prove $\Pi_R^{QR^*} \geq \Pi_R^{TQ^*}$:

$$\begin{aligned} & \Pi_R^{QR^*} \geq \Pi_R^{TQ^*} \\ \Longleftrightarrow \quad & \frac{(1-\sqrt{r})(\sqrt{r}+1)^2}{4\sqrt{r}}\frac{B^2}{Ae} + \frac{1}{2}(r+\sqrt{r})B + \frac{\sqrt{r}}{4(\sqrt{r}+1)}Ap \\ & \quad \geq \frac{-3r^2+2r+1}{8}\frac{B^2}{Ap} + \frac{3r+1}{4}B + \frac{Ap}{8} \\ \Longleftrightarrow \quad & \pi_R^{QR^*} - \pi_R^{TQ^*} = \frac{(\sqrt{r}-1)^2(r-1)}{8}\frac{B^2}{Ap} - \frac{(\sqrt{r}-1)^2}{4}B + \frac{\sqrt{r}-1}{8(\sqrt{r}+1)}Ap \geq 0 \\ \Longleftrightarrow \quad & \frac{\sqrt{r}-1}{8(\sqrt{r}+1)}Ap\Big((r-1)\frac{B}{Ap} - 1\Big)^2 \geq 0 \end{aligned}$$

$$\begin{aligned} \Pi_R^{T^*} \geq \Pi_R^{TC^*} \quad \Longleftrightarrow \quad & \frac{B}{2} + \frac{Ap}{4} + \frac{A}{2}\sqrt{\frac{p^2}{4} - \frac{Bp}{A}} \geq B + \frac{Ap}{8} \geq B + \frac{Ap}{8} - \frac{x_1^2}{2Ap} - \frac{Bx_1}{Ap} \\ \Longleftrightarrow \quad & \frac{B}{Ap} \leq \frac{1}{4} \end{aligned}$$

Case 3: $\frac{B}{Ap} \geq \frac{1}{4}$

for the manufacturer,

$$\Pi_M^{T^*} = \Pi_M^{TC^*} = \frac{Ap}{4} \geq \Pi_M^{QR^*} = \frac{Ae\big(r-(r-1)\frac{B}{Ae}\big)}{2(\sqrt{r}+1)} \geq \Pi_M^{TQ^*} = \frac{A}{4p}\big(p - \frac{(p-e)B}{Ae}\big)^2$$

for the retailer, the relationship $\Pi_R^{QR^*} \geq \Pi_R^{TQ^*} \geq \Pi_R^{TC^*} = \Pi_R^{T^*}$ can be obtained by the exactly same argument in Case 1 and Case 2 and we do not proved that here in detail. $\square$

CHAPTER 3

A Multi-Period Inventory Control Problem with Tax Consideration

3.1. Introduction

Asness (2012) argued in the *Wall Street Journal* against Warren Buffett – the once world richest man who supports raising taxes – that tax may have significant impact on investment decisions. Essentially one of Asness' key arguments is the influence of tax asymmetry. Tax asymmetry means that the taxation of profits and the compensation for losses are not symmetric (Graham and Smith (1999), Eldor and Zilcha (2004)). In fact, tax asymmetry has received considerable attention in such areas as economics and finance, see for example, Nance et al. (1993), Eldor and Zilcha (2002, 2004), and Creedy and Gemmell (2011) and the references therein. However, there is relatively limited research that incorporates the impact of tax asymmetry in the area of operations management. This paper addresses this fundamental issue in a stylized multi-period inventory control problem.

We consider a firm's inventory decisions over multiple periods in a finite horizon, which usually corresponds to a tax year. In each period, the firm can produce the product as the traditional inventory model does. The random demand is realized during the period and the sales revenue is collected. Unsold inventory is carried to the next period and the unmet demand is lost. At the end of the horizon, the firm faces a tax asymmetric problem in which she should pay a proportional tax only if her terminal accumulative profit is positive. The firm's objective is to maximize her expected terminal after-tax profit. Because tax asymmetry leads to asymmetric gains and losses, our model can be alternatively interpreted as a framework to model a loss-averse manufacturer whose gains or losses are assessed at the end of the planning horizon. We formulate the firm's problem as a stochastic dynamic programming problem and we derive the optimal policy and a number of structural properties. Also, we obtain a number of insights through these properties and numerical experiments.

In particular, we exploit the problem structures and show that an equity-dependent base-stock policy is optimal, where the equity level is the sum of the firm's current cumulative profit level and her on-hand inventory valued at the purchasing price. We also investigate the structure of the firm's optimal policy through some distinct analytical techniques. We show the structure of the optimal policy can be partially characterized by dividing the equity level into three regions. In the two extreme regions, the firm's order decision is not affected by the tax consideration; and in the middle region, the firm's optima policy will be affected by the tax asymmetry under which she chooses to order less than that in the case without tax consideration. We provide comprehensive numerical experiments to further investigate the middle region. The results show that the optimal order quantity first decreases and then increases in the equity level, revealing a "V"-shaped structure. The numerical examples also show that there might be significant profit loss if the true optimal policy is otherwise replaced by the optimal policy without considering tax.

We summarize our main contribution of this paper as follows. (1) We show that a state-dependent base stock policy is optimal for the proposed multi-period inventory control problem under tax asymmetry. (2) We prove the fundamental insight that in each period, there exists a period-dependent equity interval, in which the firm should order less than the optimal quantity without considering tax; but the firm should order the same quantity when her equity level is outside the interval. (3) We develop some distinct analytical techniques to tackle the inherent difficulty caused by the model formulation. (4) Our model and results can be easily adopted in more general settings such as loss-aversion that is valuated by the terminal profit.

The rest of this article is organized as follows. In the next section, we review related literature. In Section 3.3, we introduce the inventory control problem with tax consideration. Section 3.4 presents a number of structural properties that lead to characterization of the optimal policy. In this section, we also show how tax asymmetry may affect the optimal decisions and profits through a series of properties. We present numerical studies in Section 3.5. This paper is concluded in Section 3.6.

3.2. Literature Review

There are tremendous efforts in studying the tax asymmetry in the fields of economics and finance, see for example, Nance et al. (1993), Eldor and Zilcha (2002, 2004), and Creedy and Gemmell (2011) and the references therein. Zilcha and Eldor (2004) consider competitive firms operating under price uncertainty when taxation is asymmetric. They show that tax asymmetry has a significant effect on firm's optimal production level and its market value in the case where risk sharing tools do not exist. Altshuler and Auerbach (1990) study the significance of tax law asymmetries by an empirical investigation. Their main focus is on how the asymmetric treatment of gain and losses by the corporate income tax affects a firm's financial structure.

In the interface between tax research and operations management, there are two streams we will review in this paper. First, there are a number of deterministic models on production and inventory decisions with tax considerations. Munson and Rosenblatt (1997), Wilhelm et al. (2005), and Li et al. (2007) develop deterministic optimization models to study the impact of the tariff rules with local content requirements on global sourcing and production decision. The latter two works are motivated by tariff structures from Japan-Singapore Economic Partnership Agreement (JSEPA) and North American Free Trade Agreement (NAFTA), respectively.

Another stream of research focus on transfer pricing with tax issues between countries. A number of papers, including Nieckels (1976), Cohen et al. (1989), Kouvelis and Gutierrez (1997), Vidal and Goetschalckx (2001), and Goetschalckx et al. (2002), consider using transfer pricing to improve the performance of global supply chain operations. Such a pricing mechanism is widely used by MNFs to charge products and services among their own divisions or subsidiaries located in different countries. In these papers, transfer prices, along with the usual production and distribution decisions, become parts of decisions, which could take advantage of differentiated corporate tax rates in different countries (for example, Vidal and Goetschalckx 2001 explicitly consider rules imposed by tax authorities). The interactions of transfer pricing and production/distribution decisions are typically formulated as non-linear mathematical models, which are then often solved by heuristics. Hsu and Zhu (2011) develop analytical models to evaluate the impact of China's export-oriented tax and tariff structures on a multinational firm's operations in China. They compare a number of supply chain policies in a two-market (one domestic and one overseas) with uncertain demands business environment. Their analysis also offer managerial insights on how supply chain structures will evolve as the firm's business environment change.

Our work is related to the sizable literature on stochastic multi-period inventory control problem. Here we review some related papers. For the objective that aims to minimize holding and penalty costs, Morton (1971) studies a myopic policy under which the order quantity has to satisfy the optimal condition for the current period. Gaver (1959) and Morse (1959) pioneer the base-stock policy which means for each period there is a state-dependent order-up-to level S below which the optimal inventory level should be replenished back to this S. Several works develop various simple approximations to derive the optimal order-up-to level S, including Johansen and Thorstenson (2008) and Bijvank and Johansen (2012). Levi et al. (2008) propose a dual-balancing policy in which the penalty costs due to lost sales is balanced with the holding costs incurred by over-ordering. Besides the cost model, Van Donselaar et al. (2013) and Bijvank (2009) derive an approximation policy for the service fill rate criterion. For a comprehensive review one lost-sales, periodic-review models, readers are referred to Bijvank et al. (2011). Compared with these traditional models, the main difference in our model is that by considering the expected after-tax terminal profit, the costs or profits are no longer additive among periods, which poses new research questions but also technical challenges.

Finally, as explained earlier, tax asymmetry can be interpreted as loss aversion, thus our work is also related to the stream of literature on inventory management with risk preferences of the decision maker. Most research works in this area employ single-period models; see, for example, Eechhoudt et al. (1995), Martínez-de-Albéniz and Simchi-Levi (2003), Lee et al. (2015), and the references therein. There are relatively limited research that consider firms' risk preferences under multi-period settings. Chen et al. (2007) propose a framework that incorporates risk aversion in multi-period inventory models. Chen and Sun (2012) extend Chen et al. (2007) to infinite horizon with risk and ambiguity aversion. They assume that "additive independence axiom" holds, i.e., the utility of the decision maker is the sum of the utilities in each period. Contrary to this assumption, in our model, the inventory manager's expected after-tax terminal profit can be interpreted as a kind of utility that depends on the sum of all realized profits in every period of the planning horizon. That is, the inventory manager's utility is a function of the total before-tax profit of the whole planning horizon, which violates the "additive independence axiom".

In this paper, we intend to study multi-period inventory problems under tax consideration with the objective to maximize the total expected after-tax profit for a certain horizon such as a tax year. Incorporating tax asymmetry of a firm's yearly taxable profit into multiperiod inventory control problems adds one additional dimension that may help us consider operations for broader business situations. We therefore expect that this study may help enrich the inventory management literature and shed some light on further integration and development of tax issues with inventory and supply chain management.

3.3. The Model

Consider a manufacturing firm facing a finite-horizon production planning problem. The planning period under consideration is a tax year. Within this planning horizon, there are multiple decision periods such as quarters, months, or weeks. We consider the single-product decision making problem in which the firm produces one product. In each period, the firm has to decide the production quantity, given the decisions and the demand realizations in the previous periods. At the end of the planning horizon, leftover inventory is salvaged at a constant, which is normalized to zero, unmet demand is lost, and the total revenue is realized. Then, based on the net profit at the end of horizon, tax is levied. Specifically, when the net profit is positive, a fixed-percentage tax is levied and no tax will be incurred if the net profit is negative – meaning there is a loss.

Suppose there are $T+1$ periods, where periods 1 through T are decision periods, and period $T+1$ is an artificial terminal period in which the firm's terminal profit is realized and tax levied. The firm has to decide the production quantity in each period t, $t = 1, \cdots, T$.

We define

t = period index, $t = 1, \cdots, T+1$;

D_t = stochastic demand in period t, D_t can be any non-stationary demand patterns, e.g., $D_t \sim F_t(\cdot)$;

x_t = net inventory level at the beginning of period t before ordering;

y_t = net inventory level at the beginning of period t after ordering but before demand realization.

We assume the following stationary cost parameters:

c = unit ordering cost;

p = unit sales price, assume that $p > c$;

h = unit holding cost.

In addition, let

$x^+ = \max\{x, 0\}$, for any real number x;

$x^- = \max\{-x, 0\}$, for any real number x.

The sequence of events in each period t, $t = 1, \cdots, T$, is as follows: (1) the period t begins with an initial inventory level x_t and a cumulative profit u_t; (2) the firm determines order quantity $y_t - x_t$, where $y_t \geq x_t$ is the order-up-to level, and the order arrives instantaneously; (3) demand D_t is realized at d_t and satisfied, unsatisfied demand is lost, leftover inventory is carried to the next period, i.e., period $t+1$, production cost, sales revenue, and holding cost are incurred.

The ending after-tax profit in period $T+1$ given initial inventory x_T and commutative profit u_T is $u_{T+1} - \tau u^+_{T+1}$, where for ease of exposition, we assume zero salvage value in the terminal period. Let (x_t, u_t) denote the state of the system in period t. Then the state transitions between period t and period $t+1$ can be written as follows:

$$x_{t+1} = (y_t - D_t)^+;$$

$$u_{t+1} = u_t + p\min\{D_t, y_t\} - c(y_t - x_t) - h(y_t - D_t)^+.$$

The transition of inventory level follows directly from the lost-sales assumption while the transition of cumulative profit can be calculated from two components, one is the cumulative profit u_t at the beginning of period t and the other is the net profit $\hat{u}_t(x_t, y_t, D_t) = p\min\{D_t, y_t\} - c(y_t - x_t) - h(y_t - D_t)^+$ that the firm earns in period t after demand is realized. Note u_t should be carried to the next period until the terminal period $T+1$.

Turning to the objective of the firm. Denote $V(x_t, u_t)$ as the optimal total expected after-tax terminal profit at the beginning of period t, given the starting inventory x_t and the starting cumulative profit u_t. Then, using the theory of Markov decision processes, for $t = 1, \cdots, T$, we have the optimality equation

$$V_t(x_t, u_t) = \max_{y_t \geq x_t} E_{D_t}\left[V_{t+1}\left((y_t - D_t)^+, u_t + p\min\{D_t, y\} - c(y_t - x_t) - h(y_t - D_t)^+\right)\right],$$

and the boundary condition $V_{T+1}(x_{T+1}, u_{T+1}) = u_{T+1} - \tau u^+_{T+1}$.

Without tax consideration in the terminal period, the above problem reduces to a classical periodic-review inventory control problem and is readily solvable. However, the asymmetric tax in the terminal period makes the inventory decision significantly complex. One obvious observation is that the decisions in periods 1 through period T are all related to the terminal profit in period $T+1$. In the next section, we will explore the properties and optimal policy for the proposed problem.

3.4. Structural Properties and Optimal Policy

In this section, we first explore the problem structures and present a number of properties. We characterize the optimal policy based on these properties. We then further investigate the structure of the optimal policy through showing a series of properties. From the expression of $\hat{u}_t(x_t, y_t, D_t)$, the following lemma is obvious.

Lemma 3.4.1. *$\hat{u}_t(x_t, y_t, D_t)$ is concave in y_t for any given x_t and D_t, $t = 1, \cdots, T$.*

The following lemma indicates that the total expected after-tax profit is increasing in the initial profit in any period, which is intuitively true.

Lemma 3.4.2. *$V_t(x_t, u_t)$ is increasing in u_t for any fixed x_t, $t = 1, \cdots, T+1$.*

Proof. We prove by induction. First, it is obvious that $V_{T+1}(x_{T+1}, u_{T+1}) = u_{T+1} - \tau(u_{T+1})^+$ is increasing in u_{T+1} for any fixed x_{T+1}. Suppose now $V_{t+1}(x_{t+1}, u_{t+1})$ is increasing in u_{t+1} for any fixed x_{t+1}. We prove this also holds for $V_t(x_t, u_t)$. For any u_t^1, u_t^2, where $u_t^1 \leq u_t^2$, by assumption, we have, for any fixed y_t, x_t, where $y_t \geq x_t \geq 0$, and D_t,

$$V_{t+1}((y_t - D_t)^+, u_t^1 + \hat{u}_t(x_t, y_t, D_t)) \leq V_{t+1}((y_t - D_t)^+, u_t^2 + \hat{u}_t(x_t, y_t, D_t)).$$

Suppose $y_t^* = \arg\min_{y_t \geq x_t} E_{D_t}\left[V_{t+1}((y_t - D_t)^+, u_t^1 + \hat{u}_t(x_t, y_t, D_t))\right]$, then we have

$$\begin{aligned} V_t(x_t, u_t^1) &= E_{D_t}\left[V_{t+1}((y_t^* - D_t)^+, u_t^1 + \hat{u}_t(x_t, y_t^*, D_t))\right] \\ &\leq E_{D_t}\left[V_{t+1}((y_t^* - D_t)^+, u_t^2 + \hat{u}_t(x_t, y_t^*, D_t))\right] \\ &\leq \max_{y_t \geq x_t} E_{D_t}\left[V_{t+1}((y_t - D_t)^+, u_t^2 + \hat{u}_t(x_t, y_t, D_t))\right] \\ &= V_t(x_t, u_t^2). \end{aligned}$$

This completes the proof. □

Lemma 3.4.3. *$V_t(x_t, u_t) \geq V_t(x_t + \Delta, u_t - \tilde{p}\Delta)$ for any $\tilde{p} \geq c$ and $\Delta \geq 0$.*

Proof. Suppose $y_t^* = \arg\min_{y_t \geq x_t + \Delta} V_t(x_t + \Delta, u_t - \tilde{p}\Delta)$, then we have

$$\begin{aligned} & V_t(x_t + \Delta, u_t - \tilde{p}\Delta) \\ &= E_{D_t}\left[V_{t+1}((y_t^* - D_t)^+, u_t - \tilde{p}\Delta + p\min\{y_t^*, D_t\} - cy_t^* + cx_t + c\Delta - h(y_t^* - D_t)^+)\right] \\ &\leq E_{D_t}\left[V_{t+1}((y_t^* - D_t)^+, u_t + p\min\{y_t^*, D_t\} - cy_t^* + cx_t - h(y_t^* - D_t)^+)\right] \\ &\leq \max_{y_t \geq x_t} E_{D_t}\left[V_{t+1}((y_t - D_t)^+, u_t + p\min\{y_t, D_t\} - cy_t + cx_t - h(y_t - D_t)^+)\right] \\ &= V_t(x_t, u_t). \end{aligned}$$

where the first inequality is due to Lemma 3.4.2 and the fact that $-(\tilde{p} - c)\Delta \leq 0$.

□

Lemma 3.4.4. *$V_t(x_t, u_t)$ is jointly concave in (x_t, u_t).*

Proof. We show the statement by induction. First, it is easy to verify that $V_{T+1}(x_{T+1}, u_{T+1}) = u_{T+1} - \tau(u_{T+1})^+$ is jointly concave in (x_{T+1}, u_{T+1}). Now suppose that the statement is true for $V_{t+1}(x_{t+1}, u_{t+1})$, $1 \leq t \leq T$. we will show that it will also hold for $V_t(x_t, u_t)$. Before showing this, we show that

$$\begin{aligned}
& V_{t+1}((y_t - D_t)^+, u_t + \hat{u}_t(x_t, y_t, D_t)) \\
= \; & V_{t+1}\left((y_t - D_t)^+, u_t + p\min\{D_t, y_t\} - c(y_t - x_t) - h(y_t - D_t)^+\right)
\end{aligned}$$

is jointly concave in (x_t, u_t, y_t) for any fixed D_t. Let

$$\Delta \triangleq \lambda(y_t^1 - D_t)^+ + (1-\lambda)(y_t^2 - D_t)^+ - \left(\lambda y_t^1 + (1-\lambda)y_t^2 - D_t\right)^+ \geq 0.$$

Note that

$$\begin{aligned}
& p\min\{D_t, \lambda y_t^1 + (1-\lambda)y_t^2\} - \Delta p \\
= \; & p\min\{D_t, \lambda y_t^1 + (1-\lambda)y_t^2\} \\
& -p\{\lambda y_t^1 - \lambda\min\{y_t^1, D_t\} + (1-\lambda)y_t^2 - (1-\lambda)\min\{y_t^2, D_t\}\} \\
& +p\{\lambda y_t^1 + (1-\lambda)y_t^2 - \min\{\lambda y_t^1 + (1-\lambda)y_t^2, D_t\}\} \\
= \; & \lambda p\min\{y_t^1, D_t\} + (1-\lambda)p\min\{y_t^2, D_t\}.
\end{aligned}$$

Then for any two triples (x_t^1, u_t^1, y_t^1) and (x_t^2, u_t^2, y_t^2), where $y_t^i \geq x_t^i, i = 1, 2$, and

$0 \leq \lambda \leq 1$, we have

$$
\begin{aligned}
& V_{t+1}\Big(\big(\lambda y_t^1 + (1-\lambda)y_t^2 - D_t\big)^+, \lambda u_t^1 + (1-\lambda)u_t^2 \\
& \qquad\qquad + \hat{u}_t(\lambda x_t^1 + (1-\lambda)x_t^2, \lambda y_t^1 + (1-\lambda)y_t^2, D_t)\Big) \\
= \; & V_{t+1}\Big(\big(\lambda y_t^1 + (1-\lambda)y_t^2 - D_t\big)^+, \lambda u_t^1 + (1-\lambda)u_t^2 + p\min\{D_t, \lambda y_t^1 + (1-\lambda)y_t^2\} \\
& \quad - c(\lambda y_t^1 + (1-\lambda)y_t^2 - \big(\lambda x_t^1 + (1-\lambda)x_t^2)\big) - h(\lambda y_t^1 + (1-\lambda)y_t^2 - D_t)^+\Big) \\
\geq \; & V_{t+1}\Big(\big(\lambda y_t^1 + (1-\lambda)y_t^2 - D_t\big)^+ + \Delta, \lambda u_t^1 + (1-\lambda)u_t^2 + p\min\{D_t, \lambda y_t^1 + (1-\lambda)y_t^2\} \\
& \quad - c(\lambda y_t^1 + (1-\lambda)y_t^2 - \big(\lambda x_t^1 + (1-\lambda)x_t^2)\big) - h(\lambda y_t^1 + (1-\lambda)y_t^2 - D_t)^+ - \Delta p\Big) \\
= \; & V_{t+1}\Big(\lambda(y_t^1 - D_t)^+ + (1-\lambda)(y_t^2 - D_t)^+, \lambda u_t^1 + (1-\lambda)u_t^2 + \lambda p\min\{y_t^1, D_t\} \\
& \qquad + (1-\lambda)p\min\{y_t^2, D_t\} - c(\lambda y_t^1 + (1-\lambda)y_t^2 - \big(\lambda x_t^1 + (1-\lambda)x_t^2)\big) \\
& \qquad\qquad - h(\lambda y_t^1 + (1-\lambda)y_t^2 - D_t)^+\Big) \\
\geq \; & V_{t+1}\Big(\lambda(y_t^1 - D_t)^+ + (1-\lambda)(y_t^2 - D_t)^+, \lambda u_t^1 + (1-\lambda)u_t^2 + \lambda p\min\{y_t^1, D_t\} \\
& \qquad (1-\lambda)p\min\{y_t^2, D_t\} - c(\lambda y_t^1 + (1-\lambda)y_t^2 - \big(\lambda x_t^1 + (1-\lambda)x_t^2)\big) \\
& \qquad\qquad - \lambda h(y_t^1 - D_t)^+ - (1-\lambda)(y_t^2 - D_t)^+\Big) \\
= \; & V_{t+1}\big(\lambda(y_t^1 - D_t)^+ + (1-\lambda)(y_t^2 - D_t)^+, \\
& \qquad \big(u_t^1 + \hat{u}_t(x_t^1, y_t^1, D_t)\big) + (1-\lambda)\,\big(u_t^2 + \hat{u}_t(x_t^2, y_2^1, D_t)\big)\,\big) \\
\geq \; & \lambda V_{t+1}\big((y_t^1 - D_t)^+, u_t^1 + \hat{u}_t(x_t^1, y_t^1, D_t)\big) \\
& \qquad + (1-\lambda)V_{t+1}\big((y_t^2 - D_t)^+, u_t^2 + \hat{u}_t(x_t^2, y_t^2, D_t)\big)
\end{aligned}
$$

where the first inequality is by Lemma 3.4.3 (note that here $\tilde{p} = p \geq c$), the second inequality is by the fact that $-h(y_t - D_t)^+$ is concave in y_t, and the last inequality is by the inductive assumption that $V_{t+1}(x_{t+1}, u_{t+1})$ is jointly concave in (x_{t+1}, u_{t+1}). Hence by definition $V_{t+1}((y_t - D_t)^+, u_t + \hat{u}(x_t, y_t, D_t))$ is jointly concave in (x_t, u_t, y_t). It follows directly that before demand realization $E_{D_t}[V_{t+1}((y_t - D_t)^+, u_t + \hat{u}(x_t, y_t, D_t))]$ is also jointly concave in (x_t, u_t, y_t). Note that

$$V_t(x_t, u_t) = \max_{y_t \geq x_t} E_{D_t}\left[V_{t+1}((y_t - D_t)^+, u_t + \hat{u}(x_t, y_t, D_t))\right]$$

and the maximization is over a convex set $\{(x_t, u_t, y_t) : y_t \geq x_t\}$, it follows that the function $V_t(x_t, u_t)$ is jointly concave in (x_t, u_t). □

Theorem 3.4.1. *A state-dependent base-stock policy is optimal for the problem, specifically, there is an optimal order-up-to level $y_t^*(v_t)$ in period t, where $v_t = u_t + cx_t$, such that*

(a) if $x_t < y_t^*(v_t)$, order $y_t^*(v_t) - x_t$, and

(b) if $x_t \geq y_t^*(v_t)$, order nothing.

Proof. Define

$$\Pi_t(v_t, y_t) = E_{D_t}\left[V_{t+1}((y_t - D_t)^+, v_t + p\min\{y_t, D_t\} - cy_t - h(y_t - D_t)^+)\right]$$

The state-dependent optimal order-up-to policy follows directly from the fact that $\Pi_t(v_t, y_t)$ is a concave function in y_t given v_t. □

Note that v_t can be interpreted as an equity level representing the sum of the cumulative profit and on-hand inventory valued at the purchasing price. We can see that for each period the optimal order-up-to level $y_t^*(x_t, u_t)$ is only related to this state variable (equity level) v_t. In what follows we will study how the optimal order-up-to level y_t^* is affected by different starting equity level v_t.

Define a sequence of concave functions and a sequence of control variable as follows: $H_{T+1}(y_{T+1}) = -cy_{T+1}$, and for $t = 1, \cdots, T$,

$$H_t(y_t) = E_{D_t}\left[p\min\{y_t, D_t\} - cy_t + (c-h)(y_t - D_t)^+ + H_{t+1}(\max\{(y_t - D_t)^+, z_{t+1}\}\right],$$

where $z_{T+1} = 0$ and for $t = 1, \cdots, T$, z_t is the smallest maximizer of $H_t(y_t)$ over $y_t \geq 0$. Note $H_t(y_t)$ is independent of the equity level v_t and that $\frac{\partial^2 H_t}{\partial y_t^2} \leq (c-h-p)f_t(y_t) \leq 0$ (because $p + h \geq c$), which implies $H_t(\cdot)$ is a concave function in y_t and the unique existence of z_t. It can be easily verified that z_t is the optimal inventory level without tax consideration, i.e., when $\tau = 0$. For ease of our analytical development, we reformulate this classical multiperiod dynamic newsvendor problem in accord with our problem as expressed in (1) as follows:

$$\hat{V}_t(x_t, u_t) = \max_{y_t \geq x_t} E_{D_t}\left[\hat{V}_{t+1}((y_t - D_t)^+, u_t + \hat{u}(x_t, y_t, D_t))\right], t = 1, \cdots, T,$$

with the terminal function $\hat{V}_{T+1}(x_{T+1}, u_{T+1}) = u_{T+1}$.

Corollary 3.4.1. $\frac{\partial V_t(x_t,u_t)}{\partial u_t}$ *is nonnegative and is decreasing in* u_t*.*

Proof. The result follows directly from Lemmata 3.4.2 and 3.4.4. □

Corollary 3.4.1 shows that the marginal value of profit u_t "already earned" is always nonnegative but is decreasing due to tax effect. The decreasing marginal value of the profit already earned is because a higher initial profit u_t leads to a higher chance of a positive terminal profit and a resulting higher tax payment. Note that without considering tax, $\frac{\partial V_t(x_t,u_t)}{\partial u_t}$ is a constant.

Lemma 3.4.5. *For every period t and for any $\tilde{p} \geq p$, it follows:*

$$\frac{\partial V_t(x_t, u_t)}{\partial u_t} \leq \frac{\partial V_t(x_t + \Delta, u_t - \tilde{p}\Delta)}{\partial u_t}. \tag{3.1}$$

Proof. We prove by induction. First, we can verify that the property holds for $T+1$. We assume it holds for $t+1 \in \{2, \cdots, T+1\}$.

Case (a): $y_t^* \geq x_t$. Sub-Case (i): $x_t + \Delta \leq y_t^*$, we have

$$\begin{aligned}
& \frac{\partial V_t(x_t, u_t)}{\partial u_t} \\
= \; & E_{D_t < y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(y_t^* - D_t, u_t + cx_t + pD_t - cy_t^* - h(y_t^* - D_t)\right) \\
& + E_{D_t \geq y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + cx_t + py_t^* - cy_t^*\right) \\
= \; & E_{D_t < y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(y_t^* - D_t, u_t - c\Delta + cx_t + c\Delta + pD_t - cy_t^* - h(y_t^* - D_t)\right) \\
& + E_{D_t \geq y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t - c\Delta + cx_t + c\Delta + py_t^* - cy_t^*\right) \\
= \; & \frac{\partial V_t(x_t + \Delta, u_t - c\Delta)}{\partial u_t} \leq \frac{\partial V_t(x_t + \Delta, u_t - \tilde{p}\Delta)}{\partial u_t},
\end{aligned}$$

where the inequality is by Corollary 3.4.1. Sub-Case (ii): $x_t + \Delta > y_t^*$. In this subcase, we further study two scenarios: (ii.1) $y_t^*(x_t + \Delta, u_t - p\Delta) < x_t + \Delta$ and (ii.2) $y_t^*(x_t + \Delta, u_t - p\Delta) \geq x_t + \Delta$. We first consider (ii.1): $y_t^*(x_t + \Delta, u_t - p\Delta) <$

$x_t + \Delta$. In this scenario, let $\Delta_1 = y_t^* - x_t$ and $\Delta_2 = \Delta - \Delta_1$, then we have

$$
\begin{aligned}
& \frac{\partial V_t(x_t, u_t)}{\partial u_t} \\
= \;& E_{D_t < y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(y_t^* - D_t, u_t + cx_t + pD_t - cy_t^* - h(y_t^* - D_t)\right) \\
& + E_{D_t \geq y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + cx_t + py_t^* - cy_t^*\right) \\
= \;& E_{D_t < y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(y_t^* - D_t, u_t + cx_t + pD_t - cy_t^* - h(y_t^* - D_t)\right) \\
& + E_{y_t^* \leq D_t \leq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + cx_t + py_t^* - cy_t^*\right) \\
& + E_{D_t \geq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + cx_t + py_t^* - cy_t^*\right) \\
= \;& E_{D_t < y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t + \Delta_1 - D_t, u_t + cx_t + pD_t - c(x_t + \Delta_1) - h(x_t + \Delta_1 - D_t)\right) \\
& + E_{y_t^* \leq D_t \leq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + cx_t + p(x_t + \Delta_1) - c(x_t + \Delta_1)\right) \\
& + E_{D_t \geq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + cx_t + p(x_t + \Delta_1) - c(x_t + \Delta_1)\right) \\
= \;& E_{D_t < y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t + \Delta_1 - D_t, u_t - c\Delta_1 + h\Delta_2 + pD_t - h(x_t + \Delta - D_t)\right) \\
& + E_{y_t^* \leq D_t \leq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + px_t + (p - c)\Delta_1\right) \\
& + E_{D_t \geq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + px_t + (p - c)\Delta_1\right) \\
\leq \;& E_{D_t < y_t^*} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t + \Delta_1 + \Delta_2 - D_t, u_t - p\Delta_2 - c\Delta_1 + h\Delta_2 + pD_t - h(x_t + \Delta - D_t)\right) \\
& + E_{y_t^* \leq D_t \leq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t + \Delta - D_t, u_t + px_t - p(x_t + \Delta - D_t) + (p - c)\Delta_1\right) \\
& + E_{D_t \geq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + px_t\right) \\
\leq \;& E_{D_t < x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t + \Delta - D_t, u_t - p\Delta + pD_t - h(x_t + \Delta - D_t)\right) \\
& + E_{D_t \geq x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + px_t\right) \\
= \;& \frac{\partial V_t(x_t + \Delta, u_t - p\Delta)}{\partial u_t} \leq \frac{\partial V_t(x_t + \Delta, u_t - \tilde{p}\Delta)}{\partial u_t},
\end{aligned}
$$

where the first inequality is by the inductive assumption and Corollary 3.4.1 and the second and the last inequalities is by Corollary 3.4.1. The last equality holds because $y_t^*(x_t + \Delta, u_t - p\Delta) \le x_t + \Delta$.

(ii.2) $y_t^*(x_t + \Delta, u_t - p\Delta) \ge x_t + \Delta$. In this scenario, it follows that

$$\frac{\partial V_t(x_t, u_t)}{\partial u_t} \le \frac{\partial V_t(x_t, u_t - (p-c)\Delta)}{\partial u_t} = \frac{\partial V_t(x_t + \Delta, u_t - p\Delta)}{\partial u_t} \le \frac{\partial V_t(x_t + \Delta, u_t - \tilde{p}\Delta)}{\partial u_t},$$

where the two inequalities are due to Corollary 3.4.1 and the equality is due to the fact that $y_t^*(x_t + \Delta, u_t - p\Delta) \ge x_t + \Delta$.

Case (b): $y_t^* < x_t$. We have

$$\begin{aligned}
& \frac{\partial V_t(x_t, u_t)}{\partial u_t} \\
= \ & E_{D_t < x_t} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t - D_t, u_t + pD_t - h(x_t - D_t)\right) + E_{D_t \ge x_t} \frac{\partial}{\partial u_{t+1}} V\left(0, u_t + px_t\right) \\
= \ & E_{D_t < x_t} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t - D_t, u_t + pD_t - h(x_t - D_t)\right) + E_{x_t \le D_t < x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V\left(0, u_t + px_t\right) \\
& + E_{D_t \ge x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + px_t\right) \\
\le \ & E_{D_t < x_t} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t + \Delta - D_t, u_t - p\Delta + pD_t - h(x_t + \Delta - D_t)\right) \\
& + E_{x_t \le D_t < x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t + \Delta - D_t, u_t - p\Delta + pD_t - h(x_t + \Delta - D_t)\right) \\
& + E_{D_t \ge x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + px_t\right) \\
= \ & E_{D_t < x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(x_t + \Delta - D_t, u_t - p\Delta + pD_t - h(x_t + \Delta - D_t)\right) \\
& + E_{D_t \ge x_t + \Delta} \frac{\partial}{\partial u_{t+1}} V_{t+1}\left(0, u_t + px_t\right) \\
\le \ & \frac{\partial V_t(x_t + \Delta, u_t - p\Delta)}{\partial u_t} \le \frac{\partial V_t(x_t + \Delta, u_t - \tilde{p}\Delta)}{\partial u_t},
\end{aligned}$$

where the first inequality is by the inductive assumption, the second inequality is by arguments similar to that of Sub-Case (ii.1) of Case (a), and the last inequality is by Corollary 3.4.1. $\square$

Lemma 3.4.5 extends the insight in Lemma 3.4.3. Lemma 3.4.3 shows that it is more profitable to hold cash rather than inventory. Lemma 3.4.5 further shows that the marginal value of holding cash is lower when part of the inventory is liquidated at the selling price p instead of the purchasing price c. At first glance, it seems that inequality (3.1) should hold for $\tilde{p} \geq c$ (in a fashion similar to Lemma 3.4.3). However, this is not true. Indeed, when $\tilde{p} \geq c$ we can see the inequality does not hold and this is illustrated in the following example.

Example 3.4.1. *Consider a two-period problem instance with $c = 1$, $p = 5$, $h = 1$, $\tau = 0.25$, $x = 7.78$ and $u = -50$, $D_1 \sim U[0, 10]$ and $D_2 \sim U[0, 10]$. Then we have*

$$\frac{\partial V_1(x_1, u_1)}{\partial u_1} = 0.93114 > \frac{\partial V_1(x_1 + 0.01, u_1 - 0.01c)}{\partial u_1} = 0.93113.$$

Lemma 3.4.6. *For any $t = 1, \cdots, T+1$, we have*

$$\frac{\partial V_t(x_t, u_t)}{\partial x_t} \leq c\frac{\partial V_t(x_t, u_t)}{\partial u_t}. \tag{3.2}$$

Proof. We consider two cases (a) $y_t^* < x_t$ and (b) $y_t^* \geq x_t$. First consider Case (a), we have

$$\begin{aligned}
\frac{\partial V_t(x_t, u_t)}{\partial x_t} - c\frac{\partial V_t(x_t, u_t)}{\partial u_t} &= E_{D_t \leq x_t}\frac{\partial V_{t+1}}{\partial x_{t+1}}(x_t - D_t, u_t + (p+h)D_t - hx_t) \\
&\quad -E_{D_t \leq x_t}(c+h)\frac{\partial V_{t+1}}{\partial u_{t+1}}(x_t - D_t, u_t + (p+h)D_t - hx_t) \\
&\quad +E_{D_t > x_t}(p-c)\frac{\partial V_{t+1}}{\partial u_{t+1}}(0, u_t + px_t) \\
&= \frac{\partial \Pi_t}{\partial y_t}(v_t, x_t) \leq 0,
\end{aligned}$$

where the inequality is due to the fact that $y_t^* < x_t$. Next, consider Case (b): $y_t^* \geq x_t$. it is obvious that

$$\frac{\partial V_t(x_t, u_t)}{\partial x_t} - c\frac{\partial V_t(x_t, u_t)}{\partial u_t} = 0.$$

This completes the proof. □

Lemma 3.4.6 shows the relationship between the marginal values of x_t and u_t in any period t. It further complements the insights about inventory and cash in Lemmata 3.4.3 and 3.4.5. Not only it is more profitable to hold cash rather than inventory, but also the marginal value of inventory is always no more than its corresponding cash value.

Lemma 3.4.7. *For $t = 1, \cdots, T+1$, suppose $x_t \geq z_t$, then*

$$\frac{\partial \hat{V}_t(x_t, u_t)}{\partial x_t} = (c + H_t'(x_t))\frac{\partial \hat{V}_t(x_t, u_t)}{\partial u_t}.$$

Proof. By definition, for $x_t \geq z_t$, we have

$$\hat{V}_t(x_t, u_t) = u_t + cx_t + H_t(x_t),$$

based on which the desired result follows. □

This Lemma demonstrates the exact relationship between the marginal values of x_t and u_t in any period t for the value function $\hat{V}_t(x_t, u_t)$ while we find a similar inequality follows for $V_t(x_t, u_t)$ instead of an equality relationship in the following important proposition.

Proposition 3.4.1. *For $t = 1, \cdots, T+1$, the following results hold:*

(a) $y_t^* \le z_t$*;*

(b) For $x_t \ge z_t$*,*

$$\frac{\partial V_t(x_t,u_t)}{\partial x_t} \le (c+H_t'(x_t))\frac{\partial V_t(x_t,u_t)}{\partial u_t}.$$

Proof. We prove (a) and (b) simultaneously by induction. First, we need to verify that (a) and (b) hold true for $t = T+1$. For (a), by definition $y_{T+1}^* = z_{T+1} = 0$. For (b), by the definition of $H_t(y_t)$, for $x_{T+1} \ge z_{T+1}$, we have $H_{T+1}'(x_{T+1}) = -c$, hence

$$\frac{\partial V_{T+1}(x_{T+1},u_{T+1})}{\partial x_{T+1}} = 0 = (c+H_{T+1}'(x_{T+1}))\frac{\partial V_{T+1}(x_{T+1},u_{T+1})}{\partial u_{T+1}}.$$

So (a) and (b) are both true for $t = T+1$.

Next, we assume that (a) and (b) hold true for $t+1 \in \{2,\cdots,T+1\}$, and we need to show that they both hold for t.

We first prove part (a): $y_t^* \le z_t$. To show $y_t^* \le z_t$, it suffices to show that $\frac{\partial \Pi_t}{\partial y_t}(v_t,z_t) \le 0$, i.e.,

$$\begin{aligned}
\frac{\partial \Pi_t}{\partial y_t}(v_t,z_t) &= \frac{\partial}{\partial y_t} E_{D_t} V_{t+1}\left((z_t-D_t)^+, v_t + p\min\{D_t,z_t\} - cz_t - h(z_t-D_t)^+\right) \\
&= E_{D_t\le z_t}\frac{\partial V_{t+1}}{\partial x_{t+1}}(z_t-D_t, v_t+(p+h)D_t-(c+h)z_t) \\
&\quad -E_{D_t\le z_t}(c+h)\frac{\partial V_{t+1}}{\partial u_{t+1}}(z_t-D_t, v_t+(p+h)D_t-(c+h)z_t) \\
&\quad +E_{D_t>z_t}(p-c)\frac{\partial V_{t+1}}{\partial u_{t+1}}(0, v_t+(p-c)z_t) \\
&\le 0.
\end{aligned}$$

We know that

$$
\begin{aligned}
& E_{D_t\le z_t}\frac{\partial V_{t+1}}{\partial x_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\\
& -E_{D_t\le z_t}(c+h)\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\\
& +E_{D_t>z_t}(p-c)\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(0,v_t+(p-c)z_t\right)\\
= \; & E_{D_t\le z_t-z_{t+1}}\frac{\partial V_{t+1}}{\partial x_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\\
& -E_{D_t\le z_t-z_{t+1}}\big(c+H'_{t+1}(z_t-D_t)\big)\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\\
& +E_{D_t\le z_t-z_{t+1}}H'_{t+1}(z_t-D_t)\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\\
& +E_{z_t-z_{t+1}\le D_t\le z_t}\Big(\frac{\partial V_{t+1}}{\partial x_{t+1}}-c\frac{\partial V_{t+1}}{\partial u_{t+1}}\Big)\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\\
& -E_{D_t\le z_t}h\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\\
& +E_{D_t>z_t}(p-c)\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(0,v_t+(p-c)z_t\right).
\end{aligned}
\tag{3.3}
$$

The first two terms of (3.3) satisfy

$$
\begin{aligned}
& E_{D_t\le z_t-z_{t+1}}\frac{\partial V_{t+1}}{\partial x_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\\
& -E_{D_t\le z_t-z_{t+1}}\big(c+H'_{t+1}(z_t-D_t)\big)\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\le 0
\end{aligned}
\tag{3.4}
$$

because of the inductive assumption of part (b). Note that $x_{t+1}=z_t-D_t\ge z_{t+1}$. In the third term of (3.3), $H'_{t+1}(z_t-D_t)\le 0$ for $D_t\le z_t-z_{t+1}$ and by Lemma 3.4.5, for any $D_t\le z_t$, we have

$$
\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(z_t-D_t,v_t+(p+h)D_t-(c+h)z_t\right)\ge\frac{\partial V_{t+1}}{\partial u_{t+1}}\left(0,v_t+(p-c)z_t\right).
$$

Hence it follows that

$$E_{D_t \le z_t - z_{t+1}} H'_{t+1}(z_t - D_t) \frac{\partial V_{t+1}}{\partial u_{t+1}} (z_t - D_t, v_t + (p+h)D_t - (c+h)z_t)$$
$$\le E_{D_t \le z_t - z_{t+1}} H'_{t+1}(z_t - D_t) \frac{\partial V_{t+1}}{\partial u_{t+1}} (0, v_t + (p-c)z_t) \,. \tag{3.5}$$

The forth term of (3.3) satisfies

$$E_{z_t - z_{t+1} \le D_t \le z_t} \left(\frac{\partial V_{t+1}}{\partial x_{t+1}} - c\frac{\partial V_{t+1}}{\partial u_{t+1}}\right) (z_t - D_t, v_t + (p+h)D_t - (c+h)z_t) \le 0 \tag{3.6}$$

because $\frac{\partial V_{t+1}}{\partial x_{t+1}} \le c\frac{\partial V_{t+1}}{\partial u_{t+1}}$ for $x_{t+1} = z_t - D_t \le z_{t+1}$ by Lemma 3.4.6. The fifth term of (3.3) satisfies

$$E_{D_t \le z_t} h \frac{\partial V_{t+1}}{\partial u_{t+1}} (z_t - D_t, v_t + (p+h)D_t - (c+h)z_t) \le E_{D_t \le z_t} h \frac{\partial V_{t+1}}{\partial u_{t+1}} (0, v_t + (p-c)z_t) \tag{3.7}$$

because of Lemma 3.4.5. Putting all the above results in (3.4), (3.5), (3.6), and (3.7) together we obtain the following result:

$$(3.3) \le \big(E_{D_t \le z_t - z_{t+1}} H'_{t+1}(z_t - D_t) - E_{D_t \le z_t} h + E_{D_t > z_t}(p-c)\big) \frac{\partial V_{t+1}}{\partial u_{t+1}} (0, v_t + (p-c)z_t) = 0,$$

where the equality holds because

$$E_{D_t \le z_t - z_{t+1}} H'_{t+1}(z_t - D_t) - E_{D_t \le z_t} h + E_{D_t > z_t}(p-c) = 0,$$

which is due to the fact that $H'_t(z_t) = 0$. The proof for part (a) is completed.

Next, we prove part (b). For $x_t \ge z_t$, from part (a), we know that $y_t^* \leq z_t$, hence $x_t \geq y_t^*$, meaning that the maximizer of $E_{D_t}[V_{t+1}((y_t - D_t)^+, u_t + \hat{u}(x_t, y_t, D_t))]$ is x_t, namely,

$$\begin{aligned} V_t(x_t, u_t) &= \max_{y_t \geq x_t} E_{D_t}\left[V_{t+1}((y_t - D_t)^+, u_t + \hat{u}(x_t, y_t, D_t))\right] \\ &= E_{D_t}\left[V_{t+1}((x_t - D_t)^+, u_t + \hat{u}(x_t, x_t, D_t))\right]. \end{aligned}$$

Then we have

$$\begin{aligned} & \frac{\partial V_t(x_t, u_t)}{\partial x_t} - (c + H_t'(x_t))\frac{\partial V_t(x_t, u_t)}{\partial u_t} \\ = \; & E_{D_t \leq x_t}\frac{\partial V_{t+1}}{\partial x_{t+1}}(x_t - D_t, u_t + (p+h)D_t - hx_t) \\ & -E_{D_t \leq x_t}(c + h + H_t'(x_t))\frac{\partial V_{t+1}}{\partial u_{t+1}}(x_t - D_t, u_t + (p+h)D_t - hx_t) \\ & +E_{D_t > x_t}(p - c - H_t'(x_t))\frac{\partial V_{t+1}}{\partial u_{t+1}}(0, u_t + px_t) \\ = \; & E_{D_t \leq x_t - z_{t+1}}(\frac{\partial V_{t+1}}{\partial x_{t+1}} - (c + H_{t+1}'(x_t - D_t))\frac{\partial V_{t+1}}{\partial u_{t+1}})(x_t - D_t, u_t + (p+h)D_t - hx_t) \\ & +E_{D_t \leq x_t - z_{t+1}}H_{t+1}'(x_t - D_t)\frac{\partial V_{t+1}}{\partial u_{t+1}}(x_t - D_t, u_t + (p+h)D_t - hx_t) \\ & +E_{x_t - z_{t+1} \leq D_t \leq x_t}(\frac{\partial V_{t+1}}{\partial x_{t+1}} - c\frac{\partial V_{t+1}}{\partial u_{t+1}})(x_t - D_t, u_t + (p+h)D_t - hx_t) \\ & -E_{D_t \leq x_t}(h + H_t'(x_t))\frac{\partial V_{t+1}}{\partial u_{t+1}}(x_t - D_t, u_t + (p+h)D_t - hx_t) \\ & +E_{D_t > x_t}(p - c - H_t'(x_t))\frac{\partial V_{t+1}}{\partial u_{t+1}}(0, u_t + px_t) \\ \leq \; & E_{D_t \leq x_t - z_{t+1}}(\frac{\partial \hat{V}_{t+1}}{\partial x_{t+1}} - (c + H_{t+1}'(x_t - D_t))\frac{\partial \hat{V}_{t+1}}{\partial u_{t+1}})(x_t - D_t, u_t + (p+h)D_t - hx_t) \\ & +0 - E_{D_t \leq x_t}(p + h - c)\bar{F}(x_t)\frac{\partial V_{t+1}}{\partial u_{t+1}}(0, u_t + px_t) \\ & +E_{D_t > x_t}(p + h - c)F(x_t)\frac{\partial V_{t+1}}{\partial u_{t+1}}(0, u_t + px_t) \\ = \; & 0, \end{aligned}$$

where the inequality is by the inductive assumption of part (b), note also that

$x_{t+1} = z_t - D_t \geq z_{t+1}$; $H'_{t+1}(z_t - D_t) \leq 0$ for $D_t \leq z_t - z_{t+1}$; and $\frac{\partial V_{t+1}}{\partial x_{t+1}} \leq c\frac{\partial V_{t+1}}{\partial u_{t+1}}$ for $x_{t+1} = z_t - D_t \leq z_{t+1}$ (Lemma 3.4.6). The last equality is by Lemma 3.4.7. The proof for part (b) is completed. □

Proposition 3.4.1 shows a fundamental insight: The optimal order quantity in each period is no more than the corresponding optimal order quantity without tax consideration. This extends the results for single-period problems (see, for example, Eldor and Zilcha 2002, 2004) to the multiperiod settings.

For notational convenience, define

$$U_T^- = -(p-c)F_T^{-1}\left(\frac{p-c}{p+h}\right),$$
$$U_T^+ = (c+h)F_T^{-1}\left(\frac{p-c}{p+h}\right).$$

In addition, define recursively $U_t^- = U_{t+1}^- - (p-c)z_t$ and $U_t^+ = U_{t+1}^+ + hz_t + c(z_t - z_{t+1})^+$ for $t = 1, \cdots, T-1$. As will be seen later, U_t^- can be interpreted as the lower threshold under which the taxation effect will disappear while U_t^+ as the upper threshold above which the tax will have no impact on the optimal ordering decision. From the definition, we can easily check that the relationships $U_t^- < U_{t+1}^-$ and $U_t^+ > U_{t+1}^+$ hold for any $t \in \{1, \cdots, T\}$.

In what follows we will present several lemmata that will be used to prove the main results in Theorem 2. These lemmata themselves have important implications. First, we study the decision problem for the last period T. The lemma will be used later in the proof of Lemma 3.4.9. Furthermore, the lemma itself offers meaningful insights into the single-period newsvendor-type problem with tax consideration.

Lemma 3.4.8. *For period T, the optimal order-up-to level $y_T^*(v_T)$ is as follows:*

(a) $y_T^*(v_T) = z_T = F_T^{-1}\left(\frac{p-c}{p+h}\right)$ *when* $v_T \leq U_T^-$ *or* $v_T \geq U_T^+$*;*

(b) $y_T^*(v_T) \leq z_T = F_T^{-1}\left(\frac{p-c}{p+h}\right)$ *when* $U_T^- \leq v_T \leq U_T^+$*.*

Proof. Given the initial state (x_T, u_T) and for different order-up-to level y_T, the total expected after-tax profit is

$$\begin{aligned}
\Pi_T(v_T, y_T) &= E_{D_T}\left\{u_{T+1} - \tau u_{T+1}^+\right\} \\
&= E_{D_T}\left\{\left[v_T + (p+h)D_T - (p+h)(D_T - y_T)^+ - (c+h)y_T\right]\right. \\
&\quad \left. -\tau\left(v_T + (p+h)D_T - (p+h)(D_T - y_T)^+ - (c+h)y_T\right)^+\right\}.
\end{aligned}$$

Taking the derivative of $\Pi_T(v_T, y_T)$ with respect to y_T, we obtain the following.

Case (i): $v_T \leq 0$:

$$\frac{\partial \Pi_T(v_T, y_T)}{\partial y_T} = \begin{cases} p - c - (p+h)Pr(D_T < y_T), & \text{if } 0 \leq y_T \leq \frac{-v_T}{p-c}; \\ (1-\tau)(p-c) - (1-\tau)(p+h)Pr(D_T < y_T) & \\ -\tau(c+h)Pr\left(D_T < \frac{(c+h)y_T - v_T}{p+h}\right), & \text{if } y_T \geq \frac{-v_T}{p-c}. \end{cases}$$

Case (ii): $v_T \geq 0$:

$$\frac{\partial \Pi_T(v_T, y_T)}{\partial y_T} = \begin{cases} (1-\tau)(p-c) - (1-\tau)(p+h)Pr(D_T < y_T), & \text{if } 0 \leq y_T \leq \frac{v_T}{c+h}; \\ (1-\tau)(p-c) - (1-\tau)(p+h)Pr(D_T < y_T) & \\ -\tau(c+h)Pr\left(D_T < \frac{(c+h)y_T - v_T}{p+h}\right), & \text{if } y_T \geq \frac{v_T}{c+h}. \end{cases}$$

Notice that $Pr(D_T < z_T) = \frac{p-c}{p+h}$, we see that $\frac{\partial \Pi_T(v_T, y_T)}{\partial y_T} \leq 0$. Recall that $\Pi_T(v_T, y_T)$ is concave, part (b) follows. Next we prove part (a). We discuss case (i) first.

Case (i): $v_T \leq 0$. First, we show that $\Pi_T(v_T, y_T)$ is differentiable in y_T at all but one point, which is $y_T = \frac{-v_T}{p-c}$. In fact, we have the left and right derivatives at $y_T = \frac{-v_T}{p-c}$ as follows.

$$\begin{aligned}
&\frac{\partial_- \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) \\
&= p - c - (p+h)Pr(D_T < \frac{-v_T}{p-c}), \\
&\frac{\partial_+ \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) \\
&= (1-\tau)(p-c) - (1-\tau)(p+h)Pr\left(D_T < \frac{-v_T}{p-c}\right) - \tau(c+h)Pr\left(D_T < \frac{-v_T}{p-c}\right) \\
&= p - c - (p+h)Pr(D_T < \frac{-v_T}{p-c}) - \tau\left[(p-c) - (p-c)Pr\left(D_T < \frac{-v_T}{p-c}\right)\right].
\end{aligned}$$

Note that

$$\frac{\partial_- \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) \geq \frac{\partial_+ \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right).$$

By setting $\frac{\partial_- \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) = 0$, we obtain the corresponding v_T as follows.

$$v_T = -(p-c)F^{-1}\left(\frac{p-c}{p+h}\right) = U_T^-.$$

Clearly, we have

$$\frac{\partial_- \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) \leq 0$$

for $v_T \leq U_T^-$ and

$$\frac{\partial_- \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) > 0$$

for $v_T > U_T^-$. Therefore when $v_T \leq U_T^-$, the optimal order-up-to level is obtained by setting $\frac{\partial \Pi_T}{\partial y_T}(v_T, y_T) = 0$ and the solution is invariable with v_T:

$$y_T^* = F_T^{-1}\left(\frac{p-c}{p+h}\right) \text{ if } v_T \leq U_T^-.$$

Case (ii): $v_T \geq 0$. Different from the case where $v_T \leq 0$, it is easy to show that $\Pi_T(v_T, y_T)$ is differentiable in y_T for all $y_T \geq 0$. By setting $\frac{\partial \Pi_T}{\partial y_T}\left(v_T, \frac{v_T}{c+h}\right) = 0$, we obtain the corresponding v:

$$v_T = (c+h)F^{-1}\left(\frac{p-c}{p+h}\right) = U_T^+.$$

Clearly, we have

$$\frac{\partial \Pi_T}{\partial y_T}\left(v_T, \frac{v_T}{c+h}\right) \leq 0$$

for $v_T \geq U_T^+$ and

$$\frac{\partial \Pi_T}{\partial y_T}\left(v_T, \frac{v_T}{c+h}\right) > 0$$

for $v_T < U_T^+$. Therefore when $v_T \geq U_T^+$, the optimal order-up-to level is obtained by setting $\frac{\partial \Pi_T}{\partial y_T}(v_T, y_T) = 0$ and the solution is invariable with v_T:

$$y_T^* = F_T^{-1}\left(\frac{p-c}{p+h}\right) \text{ if } v_T \geq U_T^+.$$

The proof for part (a) is then completed by combining the above two cases. □

Lemma 3.4.8 considers a single-period (the last-period) newsvendor-type decision problem. Obviously, it differs from the classic newsvendor problem by considering after-tax-profit objective. Furthermore, it also differs from the newsvendor problem with after-tax-profit objective because it considers the initial states that consist of the *initial inventory and profit.* As a result, it offers insights different from the single-period counterparts with tax asymmetry that assumes zero starting inventory and profit (e.g., Eldor and Zilcha 2002, 2004). Specifically, Lemma 3.4.8 shows that the impact of tax asymmetry depends on the *initial equity level*: when the equity level is sufficiently low or high, the tax asymmetry has no effect on the optimal order quantity, the manufacturer can make production decisions as if there is no tax asymmetry. When the equity level is intermediate, i.e., when it is in a certain interval, then the manufacturer's order decision is affected by the tax asymmetry. Specifically, the manufacturer becomes more conservative and orders less when her equity level is intermediate. These insights will be extended to the multi-period settings later in Theorem 2. Before presenting Theorem 2, we prove a related lemma and a corollary.

Lemma 3.4.9. *For each period $t \geq 1$, if $v_t \leq U^-_{t+1} - (p-c)z_t$ we can decompose the value function as follows,*

$$\Pi_t(v_t, y_t) = v_t + H_t(y_t) \tag{3.8}$$

$$V_t(x_t, u_t) = v_t + H_t(\max\{x_t, z_t\}) \tag{3.9}$$

and if $v_t \geq U^+_{t+1} + hz_t$ we have,

$$\Pi_t(v_t, y_t) = (1-\tau)(v_t + H_t(y_t)) \tag{3.10}$$

$$V_t(x_t, u_t) = (1-\tau)(v_t + H_t(\max\{x_t, z_t\})), \tag{3.11}$$

where y_t is the decisive inventory level for state (x_t, u_t) in period t.

Proof. The proof is by induction on t. According to Lemma 3.4.8, it is easy to verify that in period T, we have $\Pi_t(v_t, y_t) = v_t + H_t(y_t)$ and $V_t(x_t, u_t) = v_t + H_t(\max\{x_t, z_t\})$ for $v_T \leq U_T^-$; $\Pi_t(v_t, y_t) = (1-\tau)(v_t + H_t(y_t))$ and $V_t(x_t, u_t) = (1-\tau)(v_t + H_t(\max\{x_t, z_t\}))$ for $v_T \geq U_T^+$. This proves the result for T. Now assume inductively that the result holds for $t+1$. In what follows we should prove it also holds for period t.

If $v_t \leq U_t^-$, let $y_t = z_t$, then for any realized demand D_t, the equity level v_{t+1} should satisfy $v_{t+1} = v_t + p\min\{z_t, D_t\} - cz_t - h(z_t - D_t)^+ + c(z_t - D_t)^+ \leq v_t + (p-c)z_t \leq U_t^- + (p-c)z_t \leq U_{t+1}^-$. According to the assumption for period $t+1$ and the definition of H_{t+1}, for period t we have,

$$
\begin{aligned}
& \Pi_t(v_t, z_t) \\
= \;& E_{D_t}\left[V_{t+1}\left((z_t - D_t)^+, v_t + p\min\{z_t, D_t\} - cz_t - h(z_t - D_t)^+\right)\right] \\
= \;& E_{D_t}\left[v_t + p\min\{z_t, D_t\} - cz_t + (c-h)(z_t - D_t)^+ + H_{t+1}(\max\{(z_t - D_t)^+, z_{t+1}\})\right] \\
= \;& v_t + E_{D_t}\left[p\min\{z_t, D_t\} - cz_t + (c-h)(z_t - D_t)^+ + H_{t+1}(\max\{(z_t - D_t)^+, z_{t+1}\})\right] \\
= \;& v_t + H_t(z_t).
\end{aligned}
$$

Given v_t, we have $\frac{\partial \Pi_t(v_t, z_t)}{\partial y_t} = 0$ and note $\Pi_t(v_t, y_t)$ is a concave function in y_t, which leads to the implication that z_t is the global optimal solution of $\Pi_t(v_t, y_t)$ when $v_t \leq U_t^-$. Under the constraint $y_t \geq x_t$, the optimal inventory level y_t should be at least as the original inventory level, e.g., $y_t = \max\{x_t, z_t\}$ and the corresponding optimal value function $V_t(x_t, u_t) = v_t + H_t(\max\{x_t, z_t\})$. Using the similar argument, we can verify equation (3.10) and (3.11) when $v_t \geq U_t^+$:

$$\begin{aligned}
& \Pi_t(v_t, z_t) \\
= \;& E_{D_t}\big[V_{t+1}\big((z_t - D_t)^+, v_t + p\min\{z_t, D_t\} - cz_t - h(z_t - D_t)^+\big)\big] \\
= \;& E_{D_t}\big[(1-\tau)\big(v_t + p\min\{z_t, D_t\} - cz_t + (c-h)(z_t - D_t)^+ \\
& +H_{t+1}(\max\{(z_t - D_t)^+, z_{t+1}\})\big)\big] \\
= \;& (1-\tau)v_t + (1-\tau)\big(E_{D_t}\big[p\min\{z_t, D_t\} - cz_t + (c-h)(z_t - D_t)^+ \\
& +H_{t+1}(\max\{(z_t - D_t)^+, z_{t+1}\})\big]\big) \\
= \;& (1-\tau)(v_t + H_t(z_t)).
\end{aligned}$$

Note $\frac{\partial \Pi_t(v_t, z_t)}{\partial y_t} = 0$, leading to $V_t(x_t, U_t) = (1-\tau)\big(v_t + H_t(\max\{x_t, z_t\})\big)$ when $v_t \geq U_t^+$. □

Lemma 3.4.9 shows a decomposition property that extends the corresponding single-period counterpart, i.e., Lemma 3.4.8 (a), to the general multi-period settings. It says that in each period t, when the equity level is sufficiently low or high (i.e., $v_t < U_t^-$ or $v_t > U_t^+$), the manufacturer orders the optimal inventory level z_t as if there is no tax effect on the order decision, i.e., $y_t^* = z_t$.

Corollary 3.4.2. *The optimal expected after-tax profit $V_t(x_t, u_t)$ in any period t should satisfy the following,*

$$(1-\tau)(v_t + H_t(\max\{x_t, z_t\})) \leq V_t(x_t, u_t) \leq v_t + H_t(\max\{x_t, z_t\}).$$

Proof. Using mathematic induction, it is easy to show for any v_t in any period

t, it holds,

$$(1-\tau)(v_t + H_t(\max\{x_t, z_t\})) \leq V_t(x_t, u_t) \leq v_t + H_t(\max\{x_t, z_t\}),$$

which completes the proof. □

Lemma 3.4.9 provides an upper bound and a lower bound for the optimal profit under tax consideration. It shows the fact that when the initial profit is very low (very high), the terminal profit is certainly negative (positive), leading to zero (full) tax payment at the end of the horizon.

Summarizing the previous results including Proposition 1 to Lemma 3.4.9, the following theorem is now in place. It characterizes how the equity level affects the firm's optimal order-up-to policies.

Theorem 3.4.2. *For period $t = 1, \cdots, T-1$, the optimal order-up-to level $y_t^*(v_t)$ satisfies the following properties:*

(a) $y_t^*(v_t) \equiv z_t$ when $v_t \leq U_t^-$ or $v_t \geq U_t^+$;

(b) $y_t^*(v_t) \leq z_t$ when $U_t^- < v_t < U_t^+$;

(c) $y_t^*(v_t) \geq \max\{\frac{U_{t+1}^- - v_t}{p-c}, \frac{v_t - U_{t+1}^+}{h}\}$.

Proof. Part (a) and (b) can be proved by applying Proposition 3.4.1 and Lemma 3.4.9. Next we prove part (c). Let $y_1 = \frac{U_{t+1}^- - v_t}{p-c}$ in function $\Pi_t(v_t, y_t)$. For any realized D_t, we have $v_{t+1} \leq U_{t+1}^-$, then the decomposition property holds, which leads to $\Pi_t(v_t, y_1) = v_t + H_t(y_1)$. Note $y_1 \leq z_t$ and the concavity of $H_t(\cdot)$, it follows that

$$\frac{\partial \Pi_t(v_t, y_1)}{\partial y_t} = \frac{dH_t(y_1)}{dy_t} \geq 0,$$

which implies $y_t^* \geq \frac{U_{t+1}^- - v_t}{p-c}$. In the similar way, when $y_2 = \frac{v_t - U_{t+1}^+}{h} \leq z_t$, we have

$$\frac{\partial \Pi_t(v_t, y_2)}{\partial y_t} = (1-\tau)\frac{dH_t(y_2)}{dy_t} \geq 0.$$

Then it follows, $y_t^*(v_t) \geq \max\{\frac{U_{t+1}^- - v_t}{p-c}, \frac{v_t - U_{t+1}^+}{h}\}$, which completes the proof of part (c). □

Having discussed a number of important properties before presenting Theorem 2, the insights for this key result are now evident. We have shown that in the multi-period setting, in every period, the tax asymmetry affects manufacturer's optimal decision in a similar manner. Specifically, in each period t, there exist two state-dependent threshold values, i.e., $U_t^- < U_t^+$, that partially characterize the manufacturer's optimal ordering behavior. In each period t, when the manufacturer's equity level v_t is smaller (larger) than the lower (upper) threshold value U_t^- (U_t^+), the manufacturer's optimal order-up-to level is the same as that in the case without tax consideration (or more precisely without tax asymmetry). When the manufacturer's equity level v_t is between the two threshold values(i.e., $v_t \in [U_t^-, U_t^+]$), the manufacturer orders less than the corresponding optimal quantity without tax consideration. These imply that the manufacturer can make ordering decisions as if there is no tax or no tax asymmetry when faced with very low or high equity level, but not so when the equity level is somewhat intermediate. This can be explained as follows. When faced with very low equity level, the manufacturer knows that he will definitely earn a negative net profit, paying no tax at all. When faced with very high equity level, the manufacturer knows that he will definitely earn a positive profit, paying a tax that is proportional to his net profit. Note that this does not necessarily mean that the manufacturer can make ordering decisions easily when faced with low or high equity level. There are two reasons. First, the manufacturer still needs to know the threshold values (U_t^- and U_t^+) below or above which the impact of tax asymmetry will vanish. Second, the two threshold values are period-dependent, and the manufacturer has to determine the two values in each period.

Unfortunately, further characterizing the policy structure when the equity level falls in $[U_t^-, U_t^+]$ is technically intractable, if not impossible. There are two immediate reasons. First, from Theorem 2, we can easily see that the ordering policy when the equity level is in $[U_t^-, U_t^+]$ is not monotone, preventing us from using a number of well-established methods to prove monotone inventory policies. Second, further characterizing the policy structure requires the manipulation of the second and third order derivatives, which is prohibitively difficult considering the special nested structure of the value functions. One additional reason, as will be discussed in the next section, is that the policy structure when the equity level is in $[U_t^-, U_t^+]$ is rather complicated. Nevertheless, we will show in the next section through numerical experiments a number of additional insights, including the policy pattern when the equity level falls in $[U_t^-, U_t^+]$.

We close the this section by providing the following proposition that shows how tax rate τ affects the optimal value function $V_t(x_t, u_t)$. For convenience, we define $V_t(x_t, u_t, \tau)$ as the optimal expected after-tax profit in period t $(1 \leq t \leq T+1)$ with tax rate $\tau \in [0, 1)$.

Proposition 3.4.2. *The optimal expected after-tax profit $V_t(x_t, u_t, \tau)$ in any period t is decreasing in the tax rate τ.*

Proof. Let $\tau_1 \leq \tau_2$. We prove by induction. First, it is easy to check that

$V_{T+1}(x_{T+1}, u_{T+1}, \tau_1) \geq V_{T+1}(x_{T+1}, u_{T+1}, \tau_2)$. Suppose $V_{t+1}(x_{t+1}, u_{t+1}, \tau_1) \geq V_{t+1}(x_{t+1}, u_{t+1}, \tau_2)$ for any $t = 2, \cdots, T$. It follows that

$$\begin{aligned}
V_t(x_t, u_t, \tau_1) &= \max_{y_t \geq x_t} E_{D_t}\left[V_{t+1}\left((y_t - D_t)^+, u_t + \hat{u}_t(x_t, y_t, D_t), \tau_1\right)\right] \\
&\geq \max_{y_t \geq x_t} E_{D_t}\left[V_{t+1}\left((y_t - D_t)^+, u_t + \hat{u}_t(x_t, y_t, D_t), \tau_2\right)\right] \\
&= V_t(x_t, u_t, \tau_2),
\end{aligned}$$

which completes the proof. □

This property confirms the intuition that a larger tax rate leads to a lower after-tax profit. Thus far we have discussed a number of analytical results in this section. In the next section we will discuss numerical examples that offer some additional insights of the proposed problem.

3.5. Numerical Studies

In this section, we will conduct numerical studies to investigate the problem and policy structures. As will be seen later, we demonstrate that the optimal order quantity without considering tax asymmetry may depart from the true optimal order quantity significantly and the profit loss may be as large as over 10% compared with the true optimal profit. We also see from the numerical results that in the tax-affected interval $[U_t^-, U_t^+]$, the optimal order quantities show a "V"-shaped structure. We start our discussions with a single-period problem.

Example 3.5.1. *Consider a problem instance with* $\tau = 0.25$, $c = 1$, $p = 5$, $h = 1$, $x_T = 0$, $D \sim U[0, 10]$. *Using equations in Section 3.4, we can compute*

z_T, U_T^-, and U_T^+ to obtain

$$z_T = 6.67, U_T^- = -26.67, U_T^+ = 13.33.$$

For different initial cash level u_T, the optimal order-up-to levels are shown in Figure 3.1.

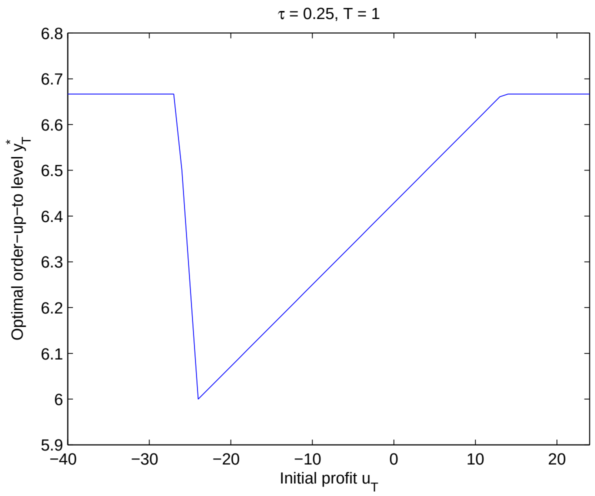

Figure 3.1: Optimal Order-up-to Levels for Different Initial Profit

In Example 3.5.1, we let the initial inventory level be zero to facilitate discussions. Recall the results in Theorem 1, the optimal order-up-to level y_T^* here solely depends on the equity value $v_T = u_T$. As we see from Figure 3.1, the initial cumulative profit u_T exerts significantly influence on y_T^* in that y_T^* is always no more than the optimal no-taxed order-up-to level z_T. Such differences may be as large as over 10% of the optimal order-up-to levels, suggesting a large deviation from the optimum may happen if tax consideration is ignored in a firm's inventory decision.

In particular, we see the optimal order-up-to level y_T^* with respect to the initial profit u_T is *not monotone*. The optimal order-up-to level shows a "V"-

shaped pattern. When U_T is small or large enough — specifically, smaller than a threshold value (U_T^-) or larger than another threshold value (U_T^+) — the optimal order-up-to level is indifferent from z_T regardless of tax rates. This is because when the initial profit is negative (positive) enough, the firm will have negative (positive) before-tax profit with probability one. Thus, under these two situations and in terms of optimal decisions, the tax effect is virtually eliminated. In addition, when both negative and positive before-tax profits are possible, i.e., when the initial profit is between the two threshold values, y_T should first decreases to a turning point U_T^0 and then increases to z_T. This observation is not intuitive because it does not necessarily mean the firm will order more aggressively when her equity level increases. We explain this result as follows. When the equity level lies in the middle, the firm is unsure about whether she will pay the tax in the last period or not. As a result, she tends to order less than the optimal inventory level without tax consideration, this finding is in accord with earlier results in Eldor and Zilcha (2002, 2004) where they claim a firm who is risk averse to the profit uncertainty will order less. In other words, the firm's behavior will become more conservative when the future net earnings are more uncertain. And this uncertainty comes the greatest when the initial profit reaches a certain value U_T^0. The above discussions about the "V"-shaped policy structure in period T when the equity level falls in $[U_T^-, U_T^+]$ can be proved analytically, as shown below.

Proposition 3.5.1. *For period T, there exists a constant U_T^0, such that the optimal order-up-to level $y_T^*(v_T)$ is decreasing in v_T when $U_T^- \leq v_T \leq U_T^0$ and is increasing in v_T when $U_T^0 \leq v_T \leq U_T^+$.*

Proof. See Appendix. □

The next example illustrates Proposition 3.5.1.

Example 3.5.2. *Continue with Example 3.4.1. Using equations from the proof of Proposition 3.5.1, we can compute U_T^0, this together with the quantities given in Example 3.5.1 lead to the following results:*

$$z_T = 6.67, U_T^- = -26.67, U_T^0 = -24.00, U_T^+ = 13.33.$$

A careful investigation of Proposition 3.5.1 and Example 3.5.3 offers us some additional insights. First, notice that $U_T^0 < 0$, this is not by accident. In fact, U_T^0 is the equity level at which the firm orders the smallest quantity compared with the case without tax asymmetry. U_T^0 reflects the equity level at which the firm is faced with some kind of "even" chance to be profitable or losing. Second, the fact that the optimal order quantity first decrease and then increase in v_T reflects the probability at which the firm's terminal profit is positive (negative) is increasing (decreasing) in the initial equity level. The equity level U_T^0 is the point at which the two opposite chances becomes equally important.

Although it is technically intractable to generalize the above optimal policy structures to every decision period of the planning horizon, it holds true in general based on the intensive numerical studies we conducted. In the following provide a two-period example for illustration.

Example 3.5.3. *Consider a two-period problem instance with $c = 1$, $p = 5$, $h = 1$, $x = 0$, $D_1 \sim U[0, 10]$ and $D_2 \sim U[0, 10]$. The optimal order-up-to levels in the first decision period (i.e., $t = 1$) for two different tax rates $\tau = 0.25$ and 0.4 are shown in Figure 3.2.*

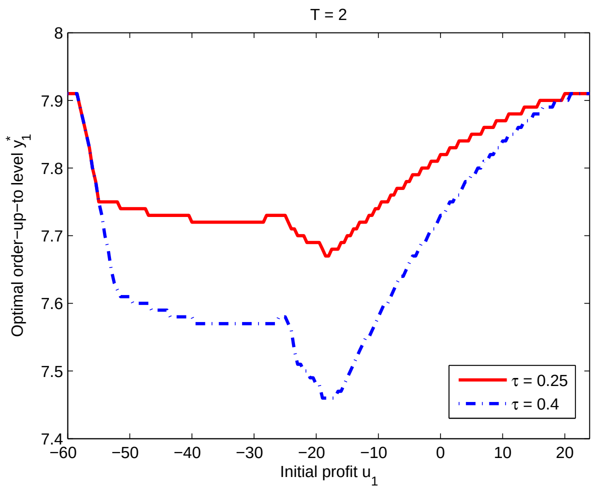

Figure 3.2: Optimal Order-up-to Level for Different Initial Profit

For both cases with different tax rates $\tau = 0.25$ and 0.4, in this two-period setting we observe the expected policy properties as Theorem 2 and Proposition 3.5.1 claim. In particular, there is a threshold value U_1^- (U_1^+), under (above) which the firm will behave as if there is no tax effect, irrespective of the demand realizations; when $U_1^- < v_1 < U_1^+$, the optimal order-up-to level is nearly decreasing first to a certain level U_1^0 and then increases to the non-taxed inventory level $z_1 = 7.91$, showing a quasi-"V"-shaped pattern similar to the single-period setting. Note that the turning point U_1^0 is also below 0 which means only in the case where negative initial profit occurs could the optimal order-up-to level decrease in v_1. Also, U_1^0 lies near a certain level v_1 such that $v_2 = v_1 + (p-c)y_1^*(v_1) = U_2^-$, which represents the turning point happens when the firm will always incur a tax cost in period T. All the above observations are consistent with the results for one-period problems and can be explained as the firm's conservative ordering behavior under demand uncertainty and initial loss. This is in contrast with the increasing property of y_1^* when v_1 is larger than U_1^0. However, we note the decreasing property is not entirely monotone because there exists a small interval of v_1 within which the optimal inventory level y_1^* increases mildly. This small abnormality could be attributed to the second period's inventory constraint where y_2^* cannot be reached when the second-period demand is rather low, e.g., $x_2 = y_1 - D_1 \geq y_2^*$. For such x_2 and in the function $V_2(x_2, u_2)$, the marginal value of x_2 and u_2 is not decreasing as that without the constraint $y_2 \geq x_2$. Consistent from the pervious statement, we can see from this fact that the optimal policy is not perfectly "V"-shaped, which certainly hinders us from further characterizing the policy property. Furthermore, we find faced with a larger initial profit, the firm becomes more aggressive when ordering until the optimal ordering level increases back to z_1. However, In addition, Figure 3.2 shows when the tax rate increases, e.g., from $\tau = 0.25$ to $\tau = 0.4$, the optimal order-up-to level y_1^* should weakly decreases for every given initial profit. This observation indicates the fact that the firm becomes more conservative and order less when the degree of tax asymmetry becomes higher. In this sense, it is more important for the firm to adjust the traditional inventory policy to explicitly consider the impact of tax.

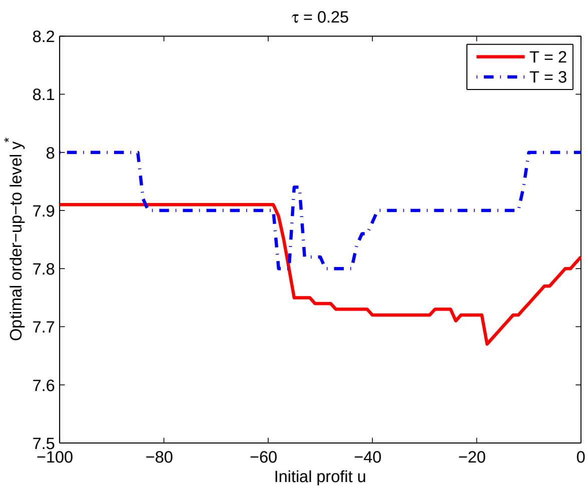

Figure 3.3: Optimal Order-up-to Level for Different Length of Horizon

Next, we study how the length of horizon affects the optimal solution. Figure 3.3 shows the optimal order-up-to level for the inventory problem with horizon three and horizon two respectively. We see that with the same initial profit u and tax-rate $\tau = 0.25$, in both cases the order-up-to level deviates from the the no-taxed inventory level z, and this deviation shares a similar shape but is much significant in the former case, e.g., the deviation from 7.9 to 7.6 versus that from 8 to 7.8. The reason behind this observation is straightforward since when the planning horizon is long, the impact caused by the terminal tax asymmetry effect will alleviate and consequently the firm opts to choose the optimal inventory position close to the non-taxed level z. This suggests that it becomes more imperative for the firm to consider tax asymmetry when she is closer to the end of the planning horizon.

Finally, we study how the optimal profit changes with respect to the initial profit. First, consider the problem in period T. Taking the first order derivative of the optimal expected profit function $V_T(x_T, u_T)$ with respect to u_T gives the following expression:

$$\frac{\partial V_T(x_T,u_T)}{\partial u_T} = \begin{cases} 1, & \text{if } v_T \leq U^- \text{ and } y_T^* \geq x_T, \\ 1-\tau, & \text{if } v_T \geq U_T^+ \text{ and } y_T^* \geq x_T, \\ 1-\tau \bar{F}_T\left(\frac{(c+h)y_T^*-v_T}{p+h}\right), & \text{if } U_T^- \leq v_T \leq U_T^+ \text{ and } y_T^* \geq x_T. \end{cases}$$

For simplicity, we assume the initial inventory level is zero (similar insights can be obtained for a positive initial inventory level). It is clear that without tax consideration (i.e., tax rate $\tau = 0$), a unit increase of the initial profit u_T will lead to a unit increase of the expected profit $V_T(x_T, u_T)$. When the tax rate is positive, we see from the expression of $\frac{\partial V_T(x_T,u_T)}{\partial u_T}$ that the tax asymmetry has unequal impacts on the expected profit for different initial equity levels. When $v_T \leq U_T^-$, that is, when the initial equity level is small enough, the final before-tax profit is definitely negative, therefore there is no tax payment and a unit increase of the initial fund will lead to a unit increase of the expected after-tax profit. Under another extreme situation, when $v_T \geq U_T^+$, that is, when the initial equity is large enough, then the pre-tax profit is always positive, and thus a fixed portion τ of the total before-tax profit has to be deducted for tax payment. Therefore, a unit increase of the initial profit will lead to a $1 - \tau$ increase of the expected after-tax profit. When v_T is at an intermediate level, tax has moderate effect on expected after-tax profit. This can be validated from the relationship that $1 - \tau \leq \frac{\partial V_T(x_T, u_T)}{\partial u_T} \leq 1$ for any v_T. It is in these "moderate" cases that tax asymmetry has the largest effect on the after-tax profit compared with the ordering behavior when tax is ignored. Because in such a region, the tax payment is dependent on the values of the initial profit and the optimal order-up-to level has the most significant deviation from the ordering level without tax consideration. The following example demonstrate the these discussions.

Example 3.5.4. *Continue with Example 3.4.1. We now vary the initial profit level and consider the percentage improvement of the true optimal ordering policy compared with the policy without considering tax asymmetry. The results are shown in Figure 3.4.*

We see from Figure 3.4 that the profit difference may be as large as over 10% of the optimal profits. This vividly shows that tax asymmetry may have significant impact on the firm's operational performance. For multi-period settings, the impact of tax asymmetry on the expected after-tax profit is similar to

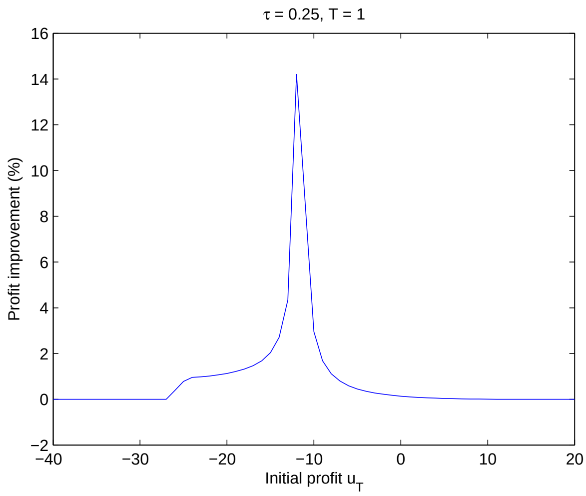

Figure 3.4: Optimal Profit Improvement for Different Initial Profit

that on ordering decisions. In particular, a firm's terminal expected after-tax profit is more severely affected by tax asymmetry when she is closer to the end the planning horizon.

3.6. Concluding Remarks

In this paper, we propose a multi-period inventory control problem which aims to maximize the expected after-tax terminal profit for a finite horizon such as a tax year. We show that a state-dependent base-stock policy is optimal. We find the firm orders less inventory when her equity level is intermediate because of the conservative behavior under demand uncertainty. We discuss a number of structural properties and provide insights on the optimal ordering decisions and profits under the consideration of tax asymmetry. We develop a series of useful properties that help partially characterize the structure of the optimal policy. We illustrate further insights through numerical examples. Notably, we find that the optimal order quantity and the optimal profit may deviate as large as over 10% from the true optimal ones if tax asymmetry is otherwise ignored.

It is easy to incoordinate a discount factor in our model and the results will hold true. Also, it is possible to extend our results to the backlogging case, in which the state-dependent base-stock policy is also optimal. However, the impact of the tax asymmetry on the optimal decisions and profits will be more complex. Yet, we believe based on our analytical and numerical studies that the basic insights will still hold in this case.

Our model has a number of natural extensions. First, as mentioned earlier, it is straightforward to apply the techniques we develop in this paper to the case of loss-averse decision maker whose utility is evaluated at the end of the planning horizon. Second, we can consider the multi-product version of the proposed problem. It would be interesting to study the interactions among different products that are otherwise irrelevant when there is no tax along this direction. Third, it would be useful to consider longer planning horizons that include multiple tax years. In this case, one has to consider the impact of loss carry back and/or forward. Finally, a number of new decisions problems arise when one combines other issues such as lead times, pricing, yield, etc.

3.7. Appendix

Proof of Proposition 3.5.1

Proof. Refering to the notation in Lemma 3.4.8 and its proof, let U_T^0 be the unique solution to

$$\frac{\partial_+ \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) = 0 \quad (v_T \leq 0),$$

where $\frac{\partial_+\Pi_T}{\partial y_T}(v_T, y_T)$ represents the right partial derivative respect to y_T. Define $y_T(v_T)$ as the optimal order-up-to level given an initial equity level v_T. Similar to the proof of Lemma 3.4.8, in the following we consider case two cases.

Case (a): $v_T \leq 0$. From the proof of Lemma 3.4.8, we see that $\Pi_T(v_T, y_T)$ is differentiable in y_T at all but one point, which is $y_T = \frac{-v_T}{p-c}$, and we have the left and right derivatives at $y_T = \frac{-v_T}{p-c}$ as follows.

$$\begin{aligned}
\frac{\partial_-\Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) &= p - c - (p+h)Pr(D_T < \frac{-v_T}{p-c}), \\
\frac{\partial_+\Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) &= p - c - (p+h)Pr(D_T < \frac{-v_T}{p-c}) \\
&\quad - \tau\left[(p-c) - (p-c)Pr\left(D_T < \frac{-v_T}{p-c}\right)\right].
\end{aligned}$$

Note that

$$\frac{\partial_-\Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) \geq \frac{\partial_+\Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right).$$

By setting $\frac{\partial_+\Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) = 0$, we obtain the corresponding v_T, which is denoted as U_T^0 as discussed earlier. Note that $U_T^0 \geq U_T^-$. Clearly, we have

$$\frac{\partial_+\Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) \leq 0$$

for $v_T \leq U_T^0$ and

$$\frac{\partial_+\Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) > 0$$

for $v_T > U_T^0$. Therefore, when $v_T \in [U_T^0, 0]$, by setting $\frac{\partial_+\Pi_T}{\partial y_T}(v_T, y_T) = 0$, we obtain the optimal order-up-to level $y_T(v_T)$. Note that

$$\frac{\partial^2 \Pi_T}{\partial y_T \partial v_T}(v_T, y_T) = \frac{\tau(c+h)}{p+h} Pr\left(D_T < \frac{(c+h)y_T - v_T}{p+h}\right) \geq 0, \quad y_T \geq \frac{-v_T}{p-c}.$$

Thus, when $v_T \in [U_T^0, 0]$, $y_T(v_T)$ is increasing in v_T. Now consider $v_T \in [U_T^-, U_T^0]$. For all such v_T, we have

$$\frac{\partial_- \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) \geq 0$$

and

$$\frac{\partial_+ \Pi_T}{\partial y_T}\left(v_T, \frac{-v_T}{p-c}\right) \leq 0.$$

Hence the optimal order up to level is

$$y_T^* = \frac{-v_T}{p-c},$$

which is decreasing in v_T.

Case (b): $v_T \geq 0$. Recall from the proof of Lemma 3.4.8 that when $v_T \leq 0$, $\Pi_T(v_T, y_T)$ is differentiable in y_T for all $y_T \geq 0$ and that the optimal order-up-to level is

$$y_T^* = F_T^{-1}\left(\frac{p-c}{p+h}\right) \text{ if } v_T \geq U_T^+.$$

When $v_T < U_T^+$, by setting $\frac{\partial \Pi_T}{\partial y_T}(v_T, y_T) = 0$, we obtain the optimal order-up-to level $y_T(v_T)$. Notice that

$$\frac{\partial^2 \Pi_T}{\partial y_T \partial v_T}(v_T, y_T) = \frac{\tau(c+h)}{p+h} Pr\left(D_T < \frac{(c+h)y_T - v_T}{p+h}\right) \geq 0, \quad y_T \geq \frac{v_T}{c+h}.$$

Thus, when $v_T \in [0, U_T^+]$, $y_T(v_T)$ is increasing in v_T.

Finally, notice that the above two cases (a) and (b) becomes invariant when $v_T = 0$. The proof is then completed by combining the above two cases. $\square$

Bibliography

Alan, Y., and Gaur, V. (2012). Operational investment and capital structure under asset based lending, Working paper, Cornell Johnson Business School.

Asness, C. S. 2012. Warren Buffett Knows That Tax Rates Matter, *Wall Street Journal* (December 17) A15.

Bijvank, M., and Johansen, S. G. 2012. Periodic review lost-sales inventory models with compound Poisson demand and constant lead times of any length, *European Journal of Operational Research* **220(1)**, 106–114.

Bijvank, M., and Vis, I. F. 2011. Lost-sales inventory theory: A review, *European Journal of Operational Research* **215(1)**, 1–13.

Burkart, M., and Ellingsen, T. (2004). In-kind finance: A theory of trade credit, *American Economic Review* **55(3)**, 569–590.

Buzacott, J. A., and Zhang, R. Q. (2004). Inventory management with asset-based financing, *Management Science* **50(9)**, 1274–1292.

Caro, F., and V. Martínez-de-Albéniz (2010). The impact of quick response in inventory-based competition, *Manufacturing & Service Operations Management* **12(3)**, 409–429.

Caro, F., and V. Martínez-de-Albéniz (2014). Fast fashion: Business model overview and research opportunities, *Retail supply chain management:*

Quantitative models and empirical studies (2nd ed.). New York: Springer (2014).

Cachon, G. P., and Swinney, R. (2009). Purchasing, pricing, and quick response in the presence of strategic consumers, *Management Science* **55(3)**, 497–511.

Cachon, G. P., and Swinney, R. (2011). The value of fast fashion: Quick response, enhanced design, and strategic consumer behavior, *Management Science* **57(4)**, 778–795.

Cai, G. G., Chen, X., and Xiao, Z. (2014). The roles of bank and trade credits: Theoretical analysis and empirical evidence, *Production and Operations Management* **23(4)**, 583–598.

Caldentey, R., and Chen, X. (2009). The role of financial services in procurement contracts, Working paper, New York University.

Chao, X., Chen, J., and Wang, S. (2008). Dynamic inventory management with cash flow constraints, *Naval Research Logistics* **55(8)**, 758–768.

Chen, X., and Cai, G. G. (2011). Joint logistics and financial services by a 3PL firm, *European Journal of Operational Research* **214(3)**, 579–587.

Chen, X., Sim, M. , Simchi-Levi, D., and Sun, P. (2007). Risk aversion in inventory management, *Operations Research* **55(5)**, 828–842.

Chen, X., Sun, P. (2012). Optimal structural policies for ambiguity and risk averse inventory and pricing models. SIAM Journal on Control and Optimization, *Operations Research Letters* **55(1)**, 133-146.

Chen, L., Kök, A. G., and Tong, J. D. (2013). The effect of payment schemes on inventory decisions: The role of mental accounting, *Management Science* **59(2)**, 436–451.

Cho, S. H., and Tang, C. S. (2013). Advance selling in a supply chain under uncertain supply and demand, *Manufacturing & Service Operations Management* **15(2)**, 305–319.

Chod, J., Rudi, N., and Van Mieghem, J. A. (2010). Operational flexibility and financial hedging: Complements or substitutes? *Management Science* **56(6)**, 1030–1045.

Choi, T. M., and Li, D., Yan, H. (2006). Quick response policy with Bayesian information updates, *European Journal of Operational Research* **170(3)**, 788–808.

Choi, T. M., and Sethi, S (2010). Innovative quick response programs: A review, *International Journal of Production Economics* **127(1)**, 1–12.

Cohen, M. A., Fisher, M., and Jaikumar, R. (1989). International manufacturing and distribution networks: a normative model framework, *Managing International Manufacturing*, North-Holland, Amsterdam, 67–93.

Creedy, J., and Gemmell, N. (2011). Corporation tax asymmetries: effective tax rates and profit shifting, *International Tax and Public Finance* **18(4)**, 422–435.

Dada, M., and Hu, Q. (2008). Financing newsvendor inventory, *Operations Research Letters* **36(5)**, 569–573.

Donohue, K. L. (2000). Efficient supply contracts for fashion goods with forecast updating and two production modes, *Management Science* **46(11)**, 1397-1411.

Eeckhoudt, L., Gollier, C., and Schlesinger, H. (1995). Risk averse (and prudent) newsboy, *Management Science* **41**(5), 786–794.

Eldor, R., and Zilcha, I. (2004). Firm's output under uncertainty and asymmetric taxation, *Economica* **71** (281), 141–153.

Eldor, R., and Zilcha, I. (2002). Tax asymmetry, production and hedging, *Journal of Economics and Business* **54(3)**, 345–356.

Eppen, G. D., and Iyer, A. V. (1997). Improved fashion buying with Bayesian updating, *Operations Research* **45(6)**, 805–819.

Fisher, M., and Raman, A. (1996). Reducing the cost of demand uncertainty through accurate response to early sales, *Operations Research* **44(1)**, 87–99.

Fisher, M., Rajaram, K., and Raman, A. (2001). Optimizing inventory replenishment of retail fashion products, *Manufacturing and Service Operations Management* **3(3)**, 230–241.

Foley, C. Fritz, Raman, A., and Nathan C. Craig. (2012). Inventory-Based Lending Industry Note, Harvard Business School Note, 612–057.

Gaver Jr, D. P. (1959). On base-stock level inventory control, *Operations Research* **7(6)**, 689–703.

Ghemawat, P., and Nueno, J. L. (2003). Zara: fast fashion, Case study, *Harvard Business School, Boston*.

Giannetti, M., Burkart, M., and Ellingsen, T. (2011). What you sell is what you lend? Explaining trade credit contracts, *Review of Financial Studies* **24(4)**, 1261–1298.

Goetschalckx, M., Vidal, C. J., and Dogan, K. (2002). Modeling and design of global logistics systems: a review of integrated strategic and tactical models and design algorithms, *European Journal of Operational Research* **143(1)**, 1–18.

Gong, X., Chao, X., and Simchi-Levi, D. (2014). Dynamic inventory control with limited capital and short-term financing, *Naval Research Logistics* **61(3)**, 184–201.

Graham, J. R., and Smith, C. W. (1999). Tax Incentives to Hedge, *Journal of Finance* **54**, 2241–2262.

Gupta, D., and Wang, L. (2009). A stochastic inventory model with trade credit, *Manufacturing and Service Operations Management* **11(1)**, 4–18.

Gurnani, H., and Tang, C. S. (1999). Note: Optimal ordering decisions with uncertain cost and demand forecast updating, *Management Science* **45(10)**, 1456–1462.

Ho, C. H., Ouyang, L. Y., and Su, C. H. (2008). Optimal pricing, shipment and payment policy for an integrated supplier-buyer inventory model with two-part trade credit, *European Journal of Operational Research* **187(2)**, 496–510.

Hsu, V., and Hu, J. (2012). Global Sourcing Decisions for a Multinational Firm With Foreign Tax Credit Planning, Working paper, CUHK Business School, The Chinese University of Hong Kong, Shatin, Hong Kong.

Hsu, V., and Zhu, K. (2011). Tax-Effective Supply Chain Decisions under China's Export-Oriented Tax Policies, *Manufacturing and Service Operations Management* **13(2)**, 163–179.

Hu, Q. J., and Sobel, M. J. (2007). Echelon base-stock policies are financially sub-optimal, *Operations research letters* **35(5)**, 561–566.

Iyer, A. V., and Bergen, M. E. (1997). Quick response in manufacturer-retailer channels, *Management Science* **43(4)**, 559–570.

Jing, B., Chen, X., and Cai, G. (2012). Equilibrium financing in a distribution channel with capital constraint, *Production and Operations Management* **21(6)**, 1090–1101.

Johansen, S. G., and Thorstenson, A. (2008). Pure and restricted base-stock policies for the lost-sales inventory system with periodic review and constant lead times, 15th International Symposium on Inventories.

Kouvelis, P., and Gutierrez, G. J. (1997). The newsvendor problem in a global market: optimal centralized and decentralized control policies for a two-market stochastic inventory system, *Management Science* **43(5)**, 571–585.

Kouvelis, P., and Zhao, W. (2011). The newsvendor problem and price-only contract when bankruptcy costs exist, *Production and Operations Management* **20(6)**, 921–936.

Kouvelis, P., and Zhao, W. (2012). Financing the newsvendor: supplier vs. bank, and the structure of optimal trade credit contracts, *Operations research* **60(3)**, 566–580.

Lariviere, M. A., and Porteus, E. L. (2001). Selling to the newsvendor: An analysis of price-only contracts, *Manufacturing and Service Operations Management* **3(4)**, 293–305.

Lau, H. S., and Lau, A. H. L. (1997). Reordering strategies for a newsboy-type product, *European Journal of Operational Research* **103(3)**, 557–572.

Lee, H., and Tang, C. S. (1997). Modelling the costs and benefits of delayed product differetiation, *Management Science* **43(1)**, 40–53.

Lee, C. Y., Li, X., Yu, M. (2015). The loss-averse newsvendor problem with supply options, *Naval Research Logistics* **62(1)**, 46–59.

Levi, R., Janakiraman, G., and Nagarajan, M. (2008). A 2-approximation algorithm for stochastic inventory control models with lost sales, *Mathematics of Operations Research* **33(2)**, 351–374.

Li, L., Shubik, M., and Sobel, M. J. (2013). Control of dividends, capital subscriptions, and physical inventories, *Management Science* **59(5)**, 1107–1124.

Li, Q., and Ha, A. Y. (2008). Reactive capacity and inventory competition under demand substitution, *IIE transactions* **40(8)**, 707–717.

Li, Y., Lim, A., and Rodrigues, B. (2007). Global sourcing using local content tariff rules, *IIE Transactions* **39(5)**, 425–437.

Lin, Y. T., and Parlaktürk, A. (2012). Quick response under competition, *Production and Operations Management* **21(3)**, 518–533.

Luo, W., and Shang, K. (2012). Integrating inventory replenishment and cash payment decisions in supply chains, Working Paper, Fuqua Business School, Duke University.

Martínez-de-Albéniz, and V., Simchi-Levi, D. (2003). Mean-variance trade-offs in supply contracts, *Naval Research Logistics* **53(7)**, 603–616.

Modigliani, F., and Miller, M. H. (1958). The cost of capital, corporation finance and the theory of investment, *The American Economic Review*, 261–297.

Morse, P. M. (1959). Solutions of a class of discrete-time inventory problems, *Operations Research* **7(1)**, 67–78.

Morton, T. E. (1971). The near-myopic nature of the lagged-proportional-cost inventory problem with lost sales, *Operations Research* **19(7)**, 1708–1716.

Munson, C. L., and Rosenblatt, M. J. (1997). The impact of local content rules on global sourcing decisions, *Production and Operations Management* **6(3)**, 277–290.

Nance, D. R., Smith, C. W., and Smithson, C. W. (1993). On the determinants of corporate hedging, *Journal of Finance* **48(1)**, 267–284.

Nieckels, L. (1976). Transfer pricing in multinational firms: a heuristic programming approach and a case study, John Wiley & Sons, New York.

Ouyang, L. Y., Chang, C. T., and Teng, J. T. (2005). An EOQ model for deteriorating items under trade credits, *Journal of the Operational Research Society* **56(6)**, 719–726.

Petersen, M. A., and Rajan, R. G. (1997). Trade credit: theories and evidence, *Review of Financial Studies* **10(3)**, 661–691.

Peter (2013). Zara' s disruptive growth strategy, *The Tycoon Playbook*.

Serel, D. A. (2012). Multi-item quick response system with budget constraint, *International Journal of Production Economics* **137(2)**, 235–249.

Shi, J., Katehakis, M. N., and Melamed, B. (2012). Cash-flow based dynamic inventory management, Working paper, Rutgers Business School.

Song, X., and Cai, X. (2006). Trade credit: theories and evidence, *International Journal of Production Economics* **103(1)**, 246–251.

Taylor, T. A. (2002). Supply chain coordination under channel rebates with sales effort effects, *Management science* **48(8)**, 992–1007.

van Donselaar, K. H., and Broekmeulen, R. A. (2013). Determination of safety stocks in a lost sales inventory system with periodic review, positive lead-time, lot-sizing and a target fill rate, *International Journal of Production Economics* **143(2)**, 440–448.

Vidal, C. J., and Goetschalckx, M. (2001). Global supply chain model with transfer pricing and transportation cost allocation, *European Journal of Operational Research* **129(1)**, 134–158.

Wilhelm, W., Liang, D., Rao, B., Warrier, D., Zhu, X., and Bulusu, S. (2005). Design of international assembly systems and their supply chains under NAFTA, *Transportation Research Part E* **41(6)**, 467–493.

Xu, X., and Birge, J. R. (2004). Joint production and financing decisions: Modeling and analysis, Working paper, The University of Chicago Graduate School of Business.

Xu, X., and Birge, J. R. (2004). Equity valuation, production, and financial planning: A stochastic programming approach, *Naval Research Logistics* **53(7)**, 641–655.

Xu, X., and Birge, J. R. (2008). Operational decisions, capital structure, and managerial compensation: A news vendor perspective, *The Engineering Economist* **53(3)**, 173–196.

Yang, S. A., and Birge, J. R. (2011). How inventory is (should be) financed: Trade credit in supply chains with demand uncertainty and costs of financial distress, Working paper, University of Chicago Booth School of Business.

Zhou, J., and Groenevelt, H. (2008). Impacts of financial collaboration in a three-party supply chain, Working paper, University of Rochester.

Zipkin, P. H. (2000). Foundations of inventory management, *New York: McGraw-Hill*.